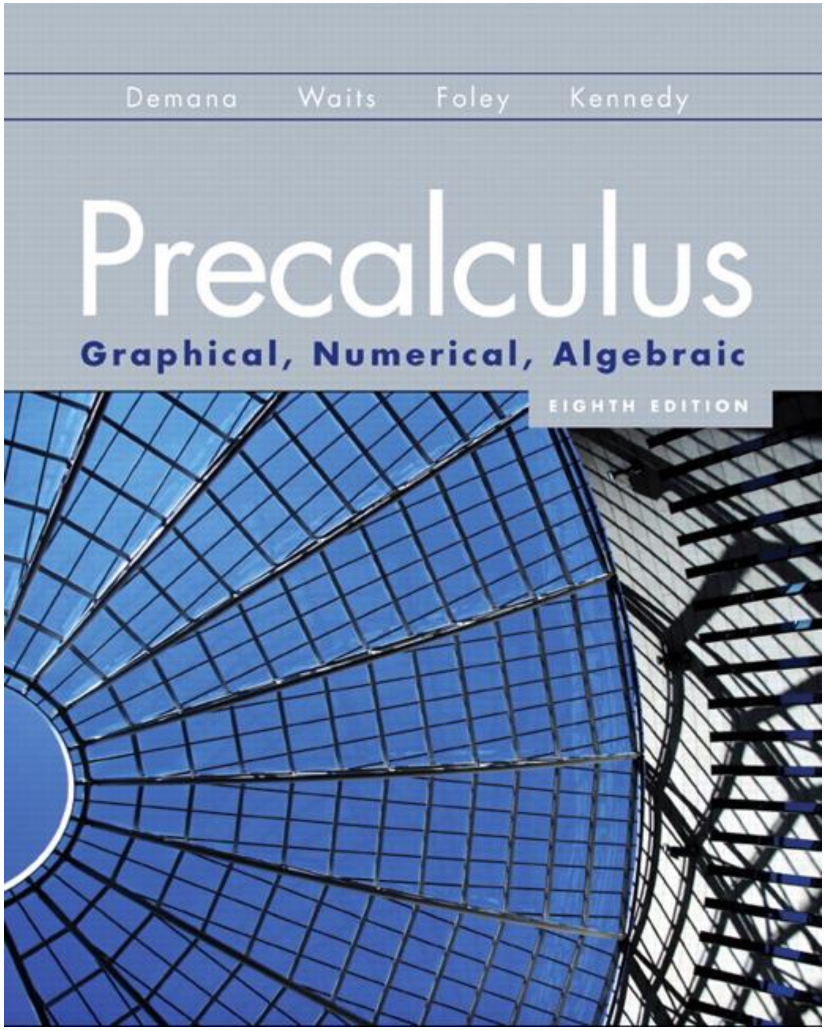
4.3

Trigonometry Extended: The Circular Functions



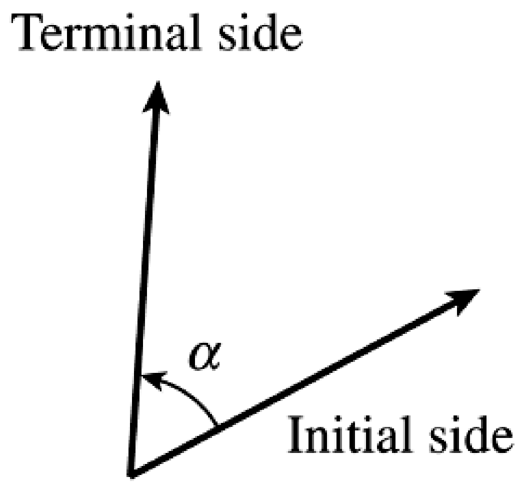
What you'll learn about

- Trigonometric Functions of Any Angle
- Trigonometric Functions of Real Numbers
- Periodic Functions
- The 16-point unit circle

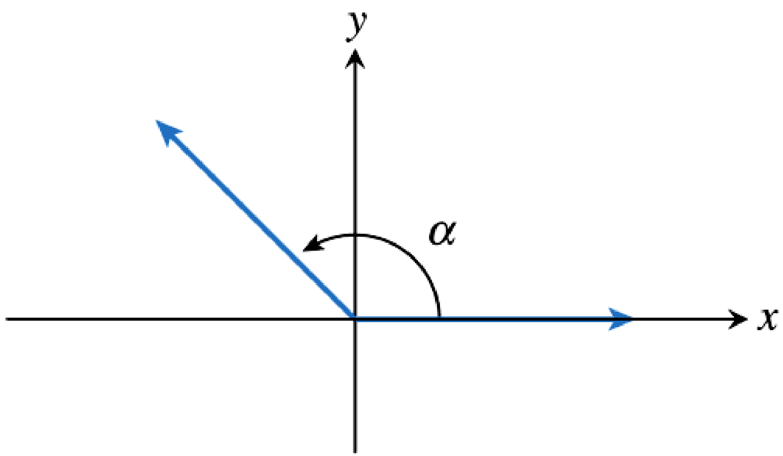
… and why

Extending trigonometric functions beyond triangle ratios opens up a new world of applications.

Initial Side, Terminal Side

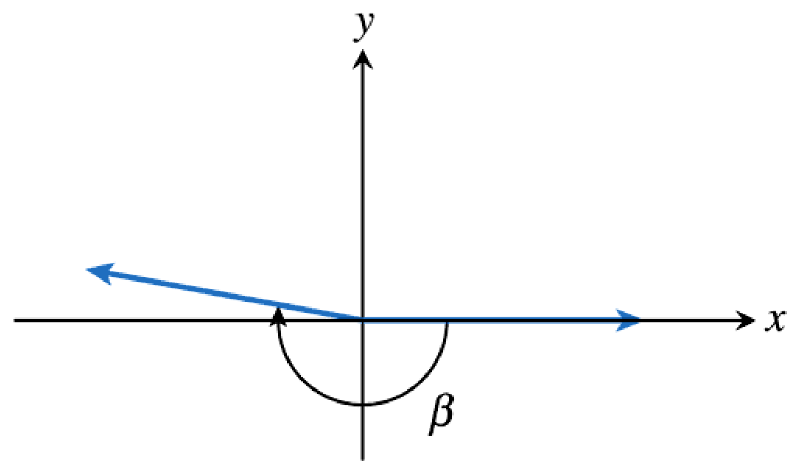


Positive Angle, Negative Angle



A positive angle
(counterclockwise)

(a)



A negative angle
(clockwise)

(b)

Coterminal Angles

Two angles in an extended angle-measurement system can have the same initial side and the same terminal side, yet have different measures. Such angles are called **coterminal angles**.

Example **Finding Coterminal Angles**

Find a positive angle and a negative angle that are coterminal with $45°$.

Example Finding Coterminal Angles

Find a positive angle and a negative angle that are coterminal with 45°.

$$\text{Add } 360^\circ: 45^\circ + 360^\circ = 405^\circ$$
$$\text{Subtract } 360^\circ: 45^\circ - 360^\circ = -315^\circ$$

Example **Finding Coterminal Angles**

Find a positive angle and a negative angle that are coterminal with $\dfrac{\pi}{6}$.

Example **Finding Coterminal Angles**

Find a positive angle and a negative angle that are coterminal with $\frac{\pi}{6}$.

$$\text{Add } 2\pi: \frac{\pi}{6} + 2\pi = \frac{13\pi}{6}$$

$$\text{Subtract } 2\pi: \frac{\pi}{6} - 2\pi = -\frac{11\pi}{6}$$

Example Evaluating Trig Functions Determined by a Point in QI

Let θ be the acute angle in standard position whose terminal side contains the point (3,5).

Find the six trigonometric functions of θ.

Example **Evaluating Trig Functions Determined by a Point in QI**

Let θ be the acute angle in standard position whose terminal side contains the point (3,5).

Find the six trigonometric functions of θ.

The distance from (3,5) to the origin is $\sqrt{34}$.

$$\sin\theta = \frac{5}{\sqrt{34}} = \frac{5\sqrt{34}}{34} \qquad \csc\theta = \frac{\sqrt{34}}{5}$$

$$\cos\theta = \frac{3}{\sqrt{34}} = \frac{3\sqrt{34}}{34} \qquad \sec\theta = \frac{\sqrt{34}}{3}$$

$$\tan\theta = \frac{5}{3} \qquad \cot\theta = \frac{3}{5}$$

Trigonometric Functions of any Angle

Let θ be any angle in standard position and let $P(x, y)$ be any point on the terminal side of the angle (except the origin). Let r denote the distance from $P(x, y)$ to the origin, i.e., let $r = \sqrt{x^2 + y^2}$. Then

$$\sin\theta = \frac{y}{r} \qquad \csc\theta = \frac{r}{y} \quad (y \neq 0)$$

$$\cos\theta = \frac{x}{r} \qquad \sec\theta = \frac{r}{x} \quad (x \neq 0)$$

$$\tan\theta = \frac{y}{x} \quad (x \neq 0) \qquad \cot\theta = \frac{x}{y} \quad (y \neq 0)$$

Evaluating Trig Functions of a Nonquadrantal Angle θ

1. Draw the angle θ in standard position, being careful to place the terminal side in the correct quadrant.
2. Without declaring a scale on either axis, label a point P (other than the origin) on the terminal side of θ.
3. Draw a perpendicular segment from P to the x-axis, determining the reference triangle. If this triangle is one of the triangles whose ratios you know, label the sides accordingly. If it is not, then you will need to use your calculator.

Evaluating Trig Functions of a Nonquadrantal Angle θ

4. Use the sides of the triangle to determine the coordinates of point *P*, making them positive or negative according to the signs of *x* and *y* in that particular quadrant.
5. Use the coordinates of point *P* and the definitions to determine the six trig functions.

Example **Evaluating More Trig Functions**

Find sin 210° without a calculator.

Example Evaluating More Trig Functions

Find sin 210° without a calculator.

An angle of 210° in standard position determines a 30°–60°–90° reference triangle in the third quadrant.

The lengths of the sides stermines the point $P\left(-\sqrt{3},-1\right)$.

The hypotenuse is $r = 2$.

$$\sin 210^\circ = \frac{y}{r} = -\frac{1}{2}$$

Example Using one Trig Ration to Find the Others

Find $\sin\theta$ and $\cot\theta$ by using the given information to construct a reference triangle.

a. $\cos\theta = -\dfrac{8}{17}$ and $\csc\theta < 0$

b. $\tan\theta = -\dfrac{1}{2}$ and $\cos\theta > 0$

Example Using one Trig Ration to Find the Others

a. $\cos\theta = -\dfrac{8}{17}$ and $\csc\theta < 0$

Since $\cos\theta < 0$ and $\csc\theta = \dfrac{1}{\sin\theta} < 0$

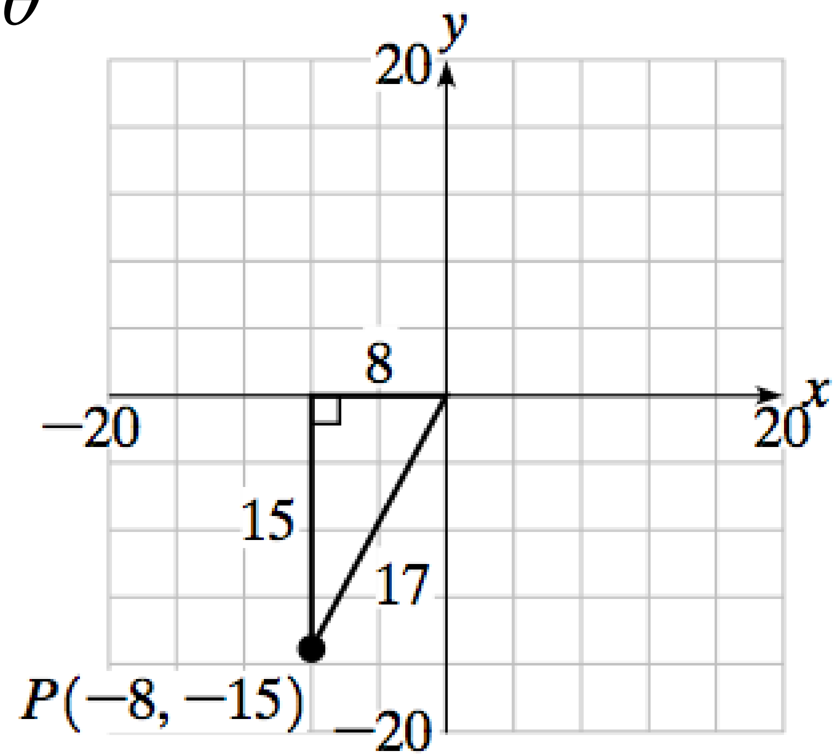
the terminal side is in QIII.

Draw a reference triangle with

$r = 17,\ x = -8.$

and $y = \sqrt{17^2 - 8^2} = -15$

$\sin\theta = -\dfrac{15}{17}$ and $\cot\theta = \dfrac{8}{15}$



Example Using one Trig Ration to Find the Others

b. $\tan\theta = -\dfrac{1}{2}$ and $\cos\theta > 0$

Since $\tan\theta < 0$ and $\cos\theta > 0,$ the terminal side is in QIV.

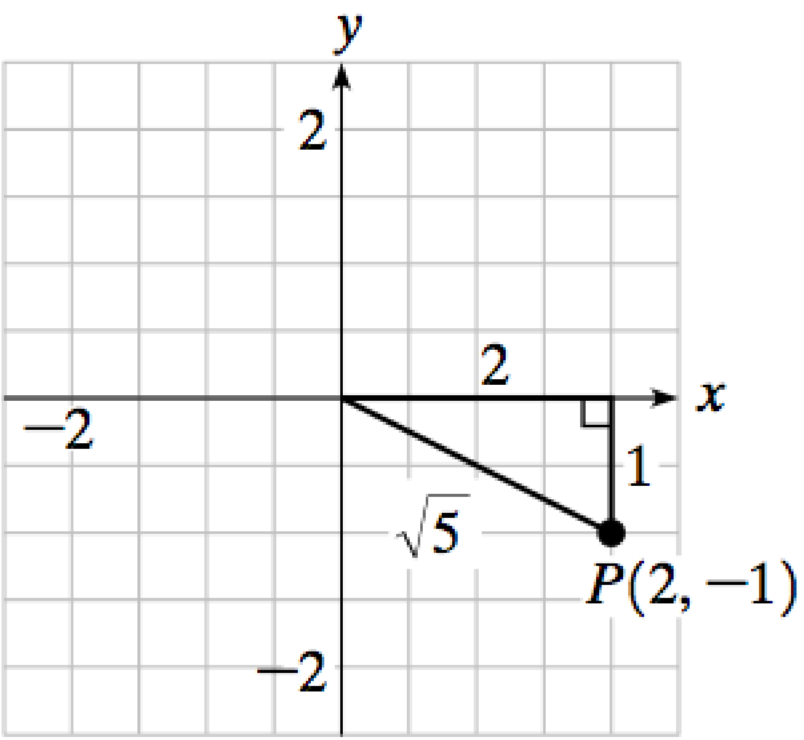
Draw a reference triangle with

$x = 2,\ y = -1.$

and $r = \sqrt{2^2 + 1^2} = \sqrt{5}$

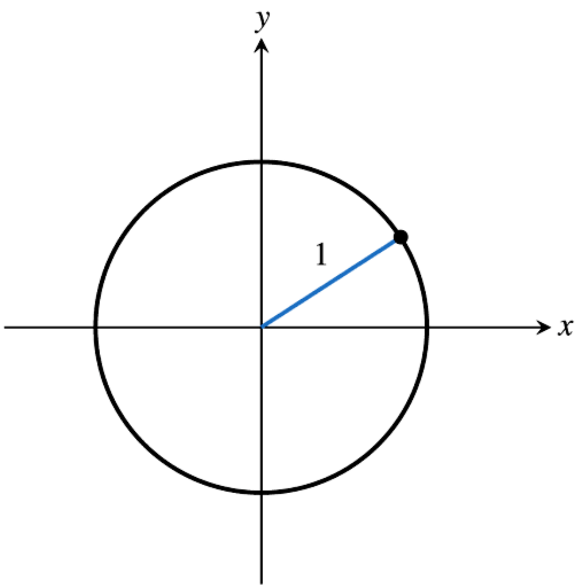
$\sin\theta = \dfrac{-1}{\sqrt{5}}$ - -0.447 and

$\cot\theta = -\dfrac{2}{1} = -2$



Unit Circle

The unit circle is a circle of radius 1 centered at the origin.



Trigonometric Functions of Real Numbers

Let t be any real number, and let $P(x, y)$ be the point corresponding to t when the number line is wrapped onto the unit circle as described above. Then

$$\sin t = y \qquad \csc t = \frac{1}{y} \quad (y \neq 0)$$

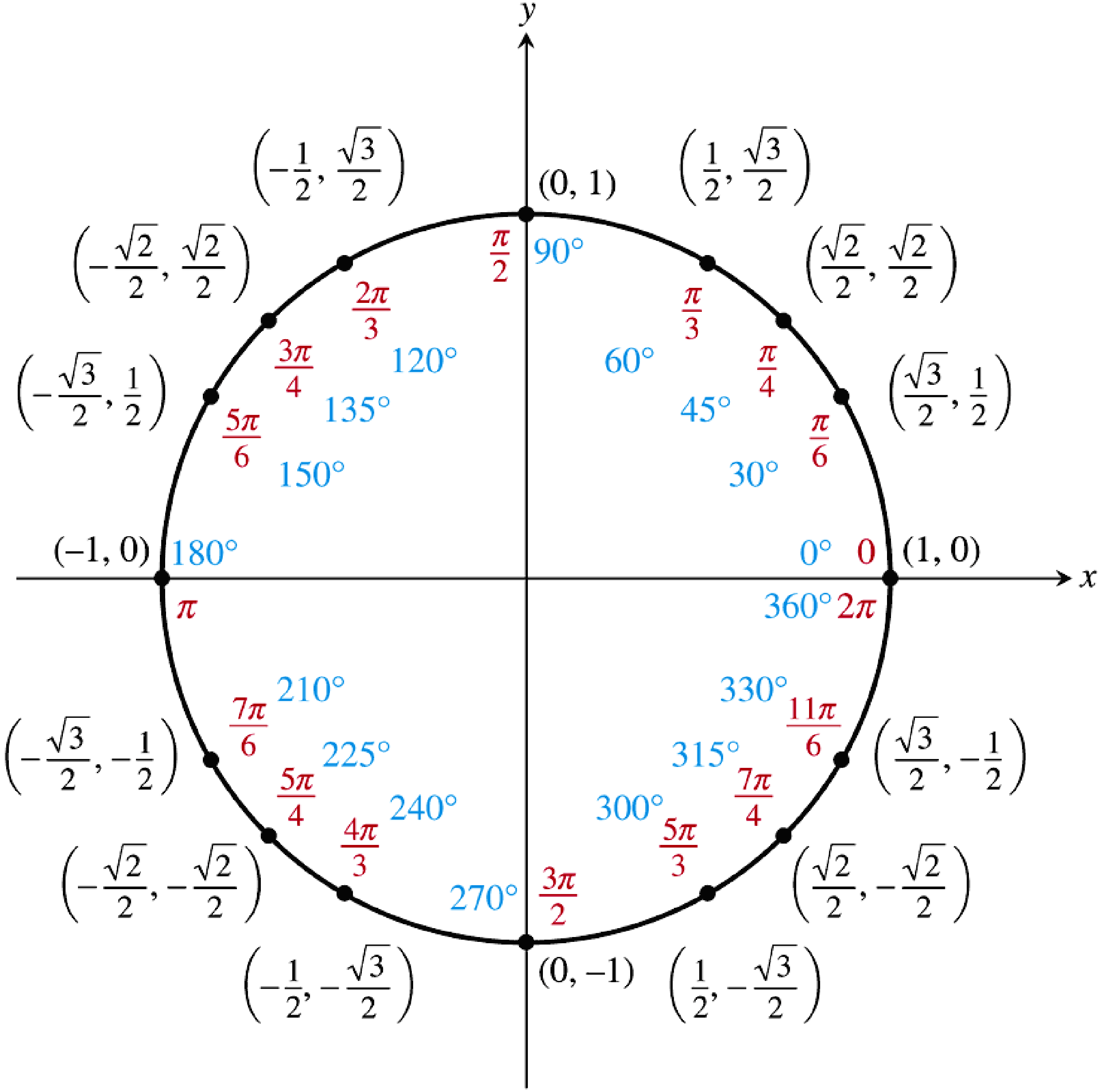
$$\cos t = x \qquad \sec t = \frac{1}{x} \quad (x \neq 0)$$

$$\tan t = \frac{y}{x} \quad (x \neq 0) \qquad \cot t = \frac{x}{y} \quad (y \neq 0)$$

Periodic Function

A function $y = f(t)$ is **periodic** if there is a positive number c such that $f(t+c) = f(t)$ for all values of t in the domain of f. The smallest such number c is called the **period** of the function.

The 16-Point Unit Circle



Quick Review

Give the value of the angle θ in degrees.

1. $\theta = \frac{2\pi}{3}$
2. $\theta = -\frac{\pi}{4}$

Use special triangles to evaluate.

3. $\cot\left(-\frac{\pi}{4}\right)$
4. $\cos\left(-\frac{7\pi}{6}\right)$

5. Use a right triangle to find the other five trigonometric functions of the acute angle θ given $\cos\theta = \frac{4}{5}$

Quick Review Solutions

Give the value of the angle θ in degrees.

1. $\theta = \dfrac{2\pi}{3}$ 120°

2. $\theta = -\dfrac{\pi}{4}$ -45°

Use special triangles to evaluate.

3. $\cot\left(-\dfrac{\pi}{4}\right)$ -1

4. $\cos\left(-\dfrac{7\pi}{6}\right)$ $-\sqrt{3}/2$

5. Use a right triangle to find the other five trigonometric functions of the acute angle θ given $\cos\theta = \dfrac{4}{5}$

$\sec\theta = 5/4,\ \sin\theta = 3/5,\ \csc\theta = 5/3,$

$\tan\theta = 3/4,\ \cot\theta = 4/3$