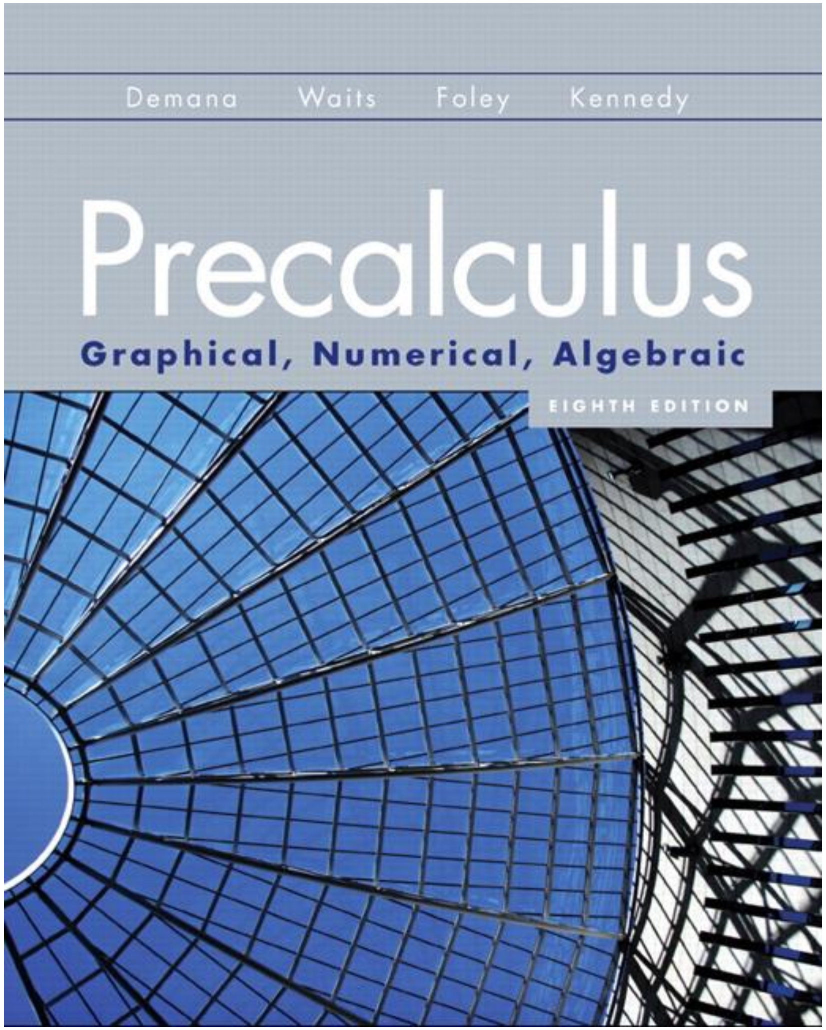
3.5

Equation Solving and Modeling



What you'll learn about

- Solving Exponential Equations
- Solving Logarithmic Equations
- Orders of Magnitude and Logarithmic Models
- Newton's Law of Cooling
- Logarithmic Re-expression

… and why

The Richter scale, pH, and Newton's Law of Cooling, are among the most important uses of logarithmic and exponential functions.

One-to-One Properties

For any exponential function $f(x) = b^x$,

- If $b^u = b^v$, then $u = v$.

For any logarithmic function $f(x) = \log_b x$,

- If $\log_b u = \log_b v$, then $u = v$.

Example **Solving an Exponential Equation Algebraically**

Solve $40(1/2)^{x/2} = 5.$

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Solve $40(1/2)^{x/2} = 5.$

$$40(1/2)^{x/2} = 5$$

$$(1/2)^{x/2} = \frac{1}{8}$$ divide by 40

$$\left(\frac{1}{2}\right)^{x/2} = \left(\frac{1}{2}\right)^{3}$$ $\frac{1}{8} = \left(\frac{1}{2}\right)^{3}$

$$x/2 = 3$$ one-to-one property

$$x = 6$$

Example **Solving a Logarithmic Equation**

Solve $\log x^3 = 3.$

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Solve $\log x^3 = 3.$

$$\log x^3 = 3$$

$$\log x^3 = \log 10^3$$

$$x^3 = 10^3$$

$$x = 10$$

Example **Solving a Logarithmic Equation**

Solve $\log(2x+1)+\log(x+3)=\log(8-2x)$

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Solve Graphically

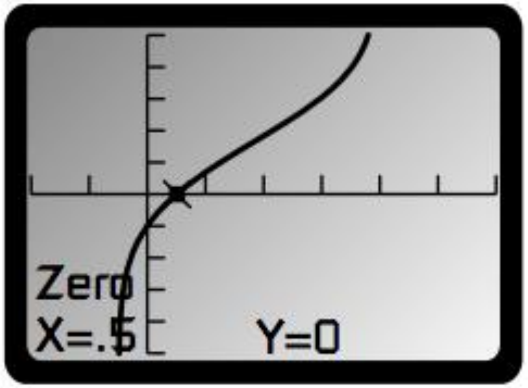
To use the *x*-intercept method, we rewrite the equation

$$\log(2x+1)+\log(x+3)-\log(8-2x)=0$$

and graph $f(x)=\log(2x+1)+\log(x+3)-\log(8-2x)$.

The *x*-intercept is $x=\dfrac{1}{2}$,

which is a solution to the equation.



[−2, 6] by [−5, 5]

Example **Solving a Logarithmic Equation**

Solve $\log(2x+1)+\log(x+3)=\log(8-2x)$

Confirm Algebraically

$$\log(2x+1)+\log(x+3)=\log(8-2x)$$

$$\log\left[(2x+1)(x+3)\right]=\log(8-2x)$$

$$(2x+1)(x+3)=(8-2x)$$

$$2x^2+9x-5=0$$

$$(2x-1)(x+5)=0 \qquad x=\frac{1}{2} \quad \text{or} \quad x=-5$$

Notice $x=-5$ is an extraneous solution. So the only solution is $x=\frac{1}{2}$.

Orders of Magnitude

The common logarithm of a positive quantity is its **order of magnitude**.

Orders of magnitude can be used to compare any like quantities:

- A kilometer is 3 orders of magnitude longer than a meter.
- A dollar is 2 orders of magnitude greater than a penny.
- New York City with 8 million people is 6 orders of magnitude bigger than Earmuff Junction with a population of 8.

Richter Scale

The Richter scale magnitude R of an earthquake is

$R = \log \frac{a}{T} + B,$ where a is the amplitude in micrometers (μm) of the vertical ground motion at the receiving station, T is the period of the associated seismic wave in seconds, and B accounts for the weakening of the seismic wave with increasing distance from the epicenter of the earthquake.

pH

In chemistry, the acidity of a water-based solution is measured by the concentration of hydrogen ions in the solution (in moles per liter). The hydrogen-ion concentration is written $[H^+]$. The measure of acidity used is **pH**, the opposite of the common log of the hydrogen-ion concentration:

$$pH = -\log [H^+]$$

More acidic solutions have higher hydrogen-ion concentrations and lower pH values.

Newton's Law of Cooling

An object that has been heated will cool to the temperature of the medium in which it is placed. The temperature T of the object at time t can be modeled by

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

for an appropriate value of k, where

T_m = the temperature of the surrounding medium,

T_0 = the temperature of the object.

This model assumes that the surrounding medium maintains a constant temperature.

Example Newton's Law of Cooling

A hard-boiled egg at temperature 100°C is placed in 15°C water to cool. Five minutes later the temperature of the egg is 55°C. When will the egg be 25°C?

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Given $T_0 = 100$, $T_m = 15$, and $T(5) = 55$.

$$T(t) = T_m + (T_0 - T_m)e^{-kt}$$

$$55 = 15 + 85e^{-5k}$$

$$40 = 85e^{-5k}$$

$$\left(\frac{40}{85}\right) = e^{-5k}$$

$$\ln\left(\frac{40}{85}\right) = -5k$$

$$k = 0.1507...$$

Now find t when $T(t) = 25$.

$$25 = 15 + 85e^{-0.1507t}$$

$$10 = 85e^{-0.1507t}$$

$$\ln\left(\frac{10}{85}\right) = -0.1507t$$

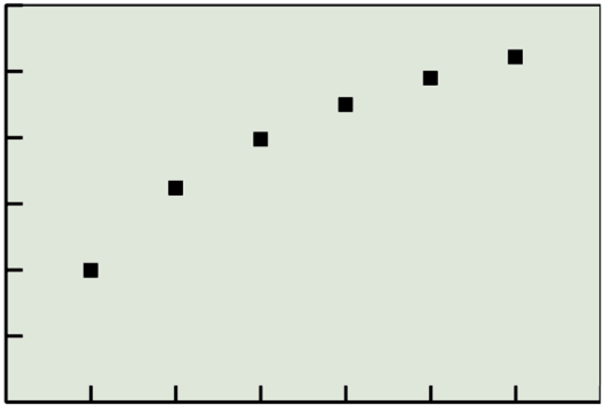
$$t = 14.2 \text{ min}.$$

Regression Models Related by Logarithmic Re-Expression

- Linear regression: $y = ax + b$
- Natural logarithmic regression: $y = a + b\ln x$
- Exponential regression: $y = a \cdot b^x$
- Power regression: $y = a \cdot x^b$

Three Types of Logarithmic Re-Expression

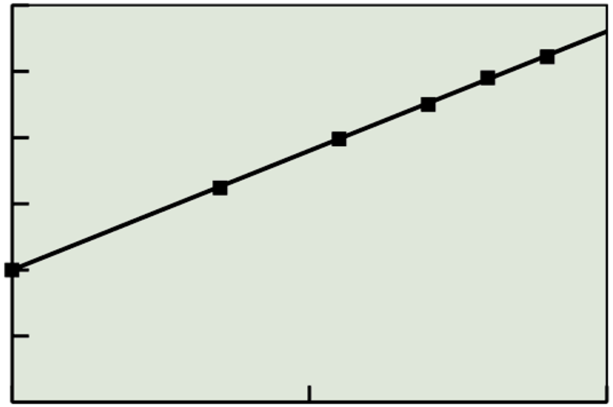
1. Natural Logarithmic Regression Re-expressed: $(x, y) \rightarrow (\ln x, y)$



[0, 7] by [0, 30]

(x, y) data

(a)



[0, 2] by [0, 30]

$(\ln x, y) = (u, y)$ data with linear regression model $y = au + b$

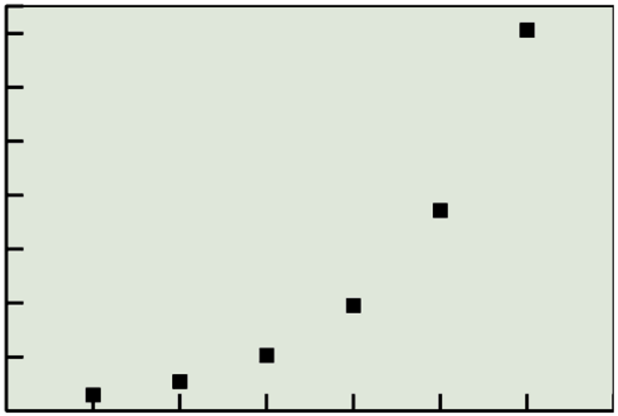
(b)

Conclusion:

$y = a \ln x + b$ is the logarithmic regression model for the (x, y) data.

Three Types of Logarithmic Re-Expression (cont'd)

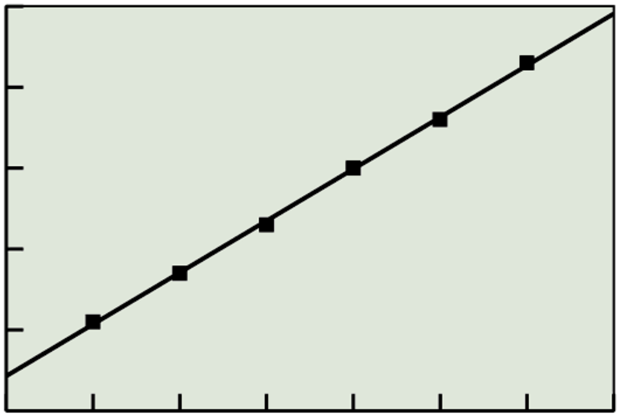
2. Exponential Regression Re-expressed: $(x, y) \rightarrow (x, \ln y)$



[0, 7] by [0, 75]

(x, y) data

(a)



[0, 7] by [0, 5]

$(x, \ln y) = (x, v)$ data with linear regression model $v = ax + b$

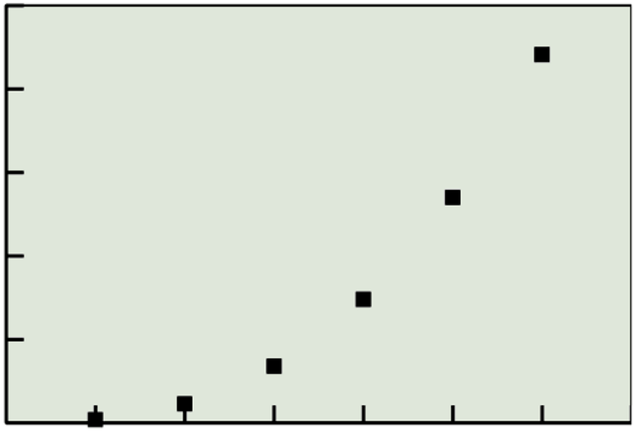
(b)

Conclusion:

$y = c(d^x)$, where $c = e^b$ and $d = e^a$, is the exponential regression model for the (x, y) data.

Three Types of Logarithmic Re-Expression (cont'd)

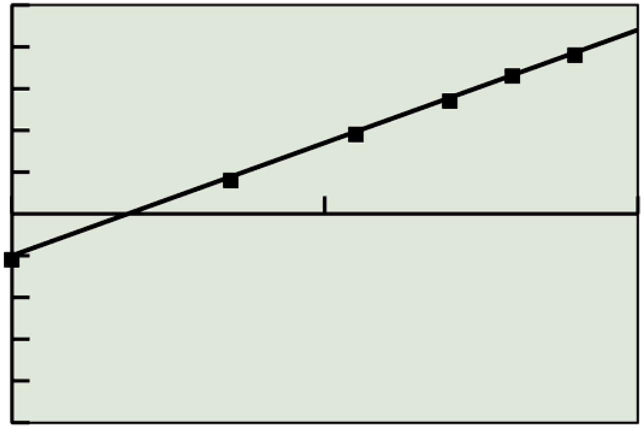
3. Power Regression Re-expressed: $(x, y) \rightarrow (\ln x, \ln y)$



[0, 7] by [0, 50]

(x, y) data

(a)



[0, 2] by [–5, 5]

$(\ln x, \ln y) = (u, v)$ data with
linear regression model
$v = au + b$

(b)

Conclusion:

$y = c(x^a)$, where $c = e^b$, is the power regression model for the (x, y) data.

Quick Review

Prove that each function in the given pair is the inverse of the other.

1. $f(x) = e^{3x}$ and $g(x) = \ln\left(x^{1/3}\right)$

2. $f(x) = \log x^2$ and $g(x) = 10^{x/2}$

Write the number in scientific notation.

3. 123,400,000

Write the number in decimal form.

4. 5.67×10^{8}

5. 8.91×10^{-4}

Quick Review Solutions

Prove that each function in the given pair is the inverse of the other.

1. $f(x) = e^{3x}$ and $g(x) = \ln\left(x^{1/3}\right)$ $f(g(x)) = e^{3\ln\left(x^{1/3}\right)} = e^{\ln(x)} = x$

2. $f(x) = \log x^2$ and $g(x) = 10^{x/2}$ $f(g(x)) = \log\left(10^{x/2}\right)^2 = \log 10^x = x$

Write the number in scientific notation.

3. 123,400,000 1.234×10^8

Write the number in decimal form.

4. 5.67×10^8 $567{,}000{,}000$

5. 8.91×10^{-4} 0.000891