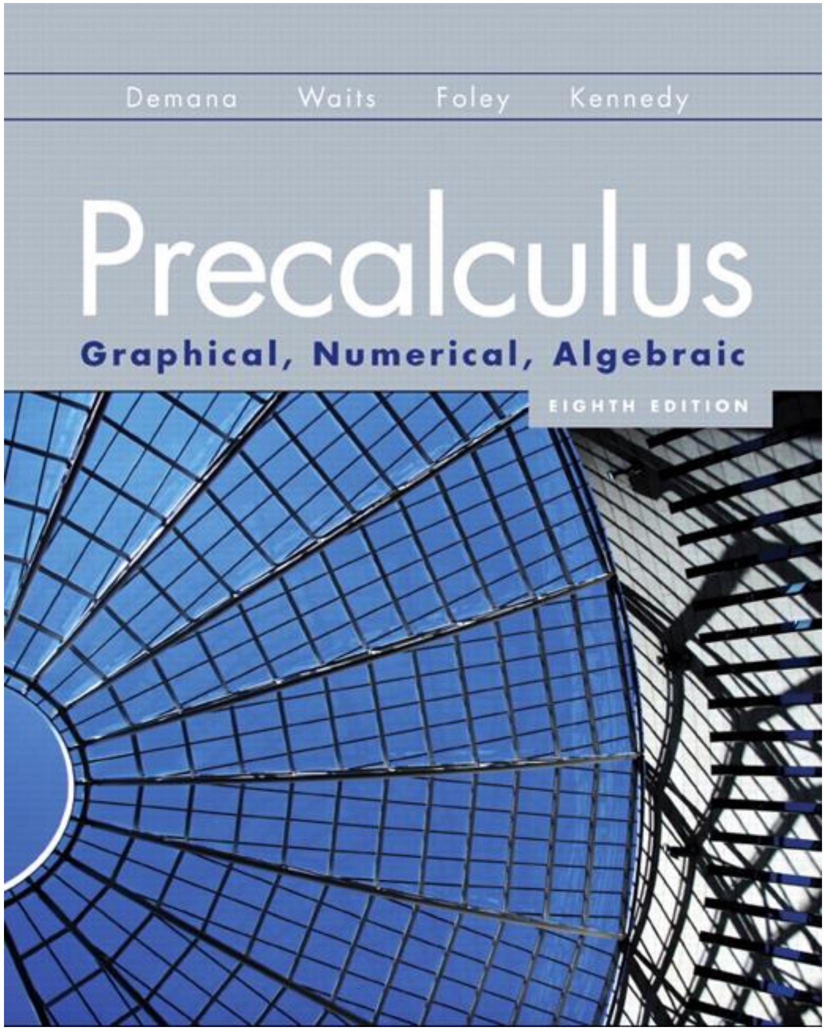
3.2

Exponential and Logistic Modeling



What you'll learn about

- Constant Percentage Rate and Exponential Functions
- Exponential Growth and Decay Models
- Using Regression to Model Population
- Other Logistic Models

… and why

Exponential functions model many types of unrestricted growth; logistic functions model restricted growth, including the spread of disease and the spread of rumors.

Constant Percentage Rate

Suppose that a population is changing at a **constant percentage rate *r***, where r is the percent rate of change expressed in decimal form. Then the population follows the pattern shown.

Time in years	Population
0	$P(0) = P_0 =$ initial population
1	$P(1) = P_0 + P_0 r = P_0(1+r)$
2	$P(2) = P(1) \cdot (1+r) = P_0(1+r)^2$
3	$P(3) = P(2) \cdot (1+r) = P_0(1+r)^3$
$\vdots$	$\vdots$
t	$P(t) = P_0(1+r)^t$

Exponential Population Model

If a population P is changing at a constant percentage rate r each year, then

$$P(t) = P_0(1+r)^t,$$

where P_0 is the initial population,
r is expressed as a decimal,
and t is time in years.

Example **Finding Growth and Decay Rates**

Tell whether the population model $P(t) = 786{,}543 \cdot 1.021^t$ is an exponential growth function or exponential decay function, and find the constant percent rate of growth.

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Tell whether the population model $P(t) = 786,543 \cdot 1.021^t$ is an exponential growth function or exponential decay function, and find the constant percent rate of growth.

Because $1 + r = 1.021$, $r = .021 > 0$.
So, P is an exponential growth function with a growth rate of 2.1%.

Example **Finding an Exponential Function**

Determine the exponential function with initial value = 10, increasing at a rate of 5% per year.

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Determine the exponential function with initial value = 10, increasing at a rate of 5% per year.

Because $P_0 = 10$ and $r = 5\% = 0.05$, the function is

$$P(t) = 10(1 + 0.05)^t \quad \text{or} \quad P(t) = 10(1.05)^t.$$

Example **Modeling Bacteria Growth**

Suppose a culture of 200 bacteria is put into a petri dish and the culture doubles every hour.

Predict when the number of bacteria will be 350,000.

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Suppose a culture of 200 bacteria is put into a petri dish and the culture doubles every hour.
Predict when the number of bacteria will be 350,000.

$400 = 200 \cdot 2$

$800 = 200 \cdot 2^2$

$\vdots$

$P(t) = 200 \cdot 2^t$

represents the bacteria population t hr after it is placed in the petri dish.
To find out when the population will reach 350,000, solve $350,000 = 200 \cdot 2^t$ for t using a calculator.

$t = 10.77$ or about 10 hours and 46 minutes.

Example Modeling U.S. Population Using Exponential Regression

Use the population data in the table to predict the population for the year 2000. Compare with the actual 2000 population of approximately 281 million.

U.S. Population, 1890–1980

Year	Population (millions)
1890	62.9
1900	76.0
1910	92.0
1920	105.7
1930	122.8
1940	131.7
1950	151.3
1960	179.3
1970	203.3
1980	226.5

Source: U. S. Bureau of the Census, *Statistical Abstract of the United States*, 2003 (Washington, D. C., 2003)

Example Modeling U.S. Population Using Exponential Regression

Let $x = 0$ represent 1890, $x = 1$ represent 1900, and so on. The year 2000 is represented by $x = 11$. Create a scatter plot of the data following the steps below.

U.S. Population, 1890–1980

Year	Population (millions)
1890	62.9
1900	76.0
1910	92.0
1920	105.7
1930	122.8
1940	131.7
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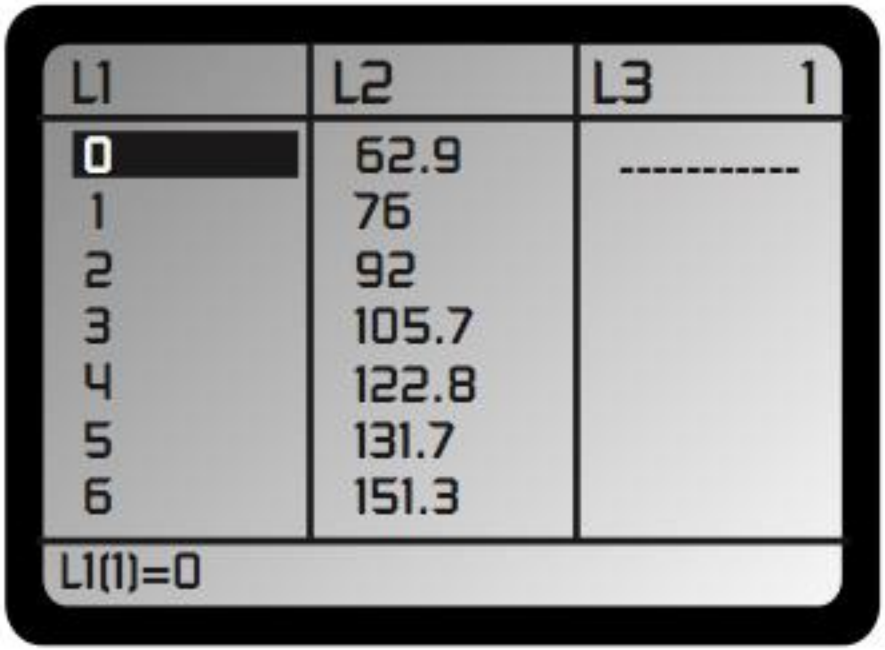
Source: U. S. Bureau of the Census, *Statistical Abstract of the United States*, 2003 (Washington, D. C., 2003)

Example **Modeling U.S. Population Using Exponential Regression**

Enter the data into the statistical memory of your grapher.

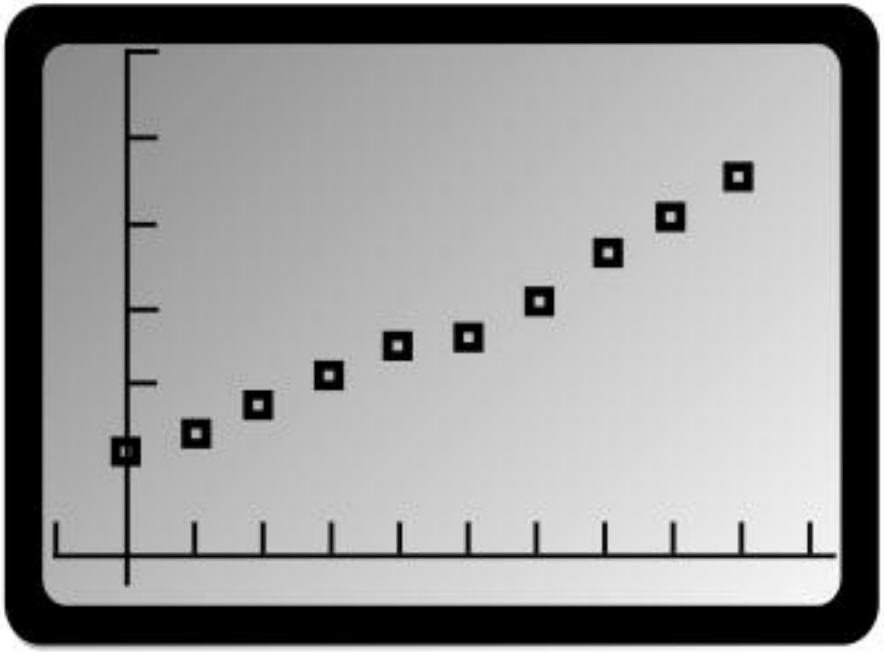


Example **Modeling U.S. Population Using Exponential Regression**

Set an appropriate window for the data, letting *x* correspond to the year and *y* to the population (in millions). Note that the year 1980 corresponds to $x = 9$.

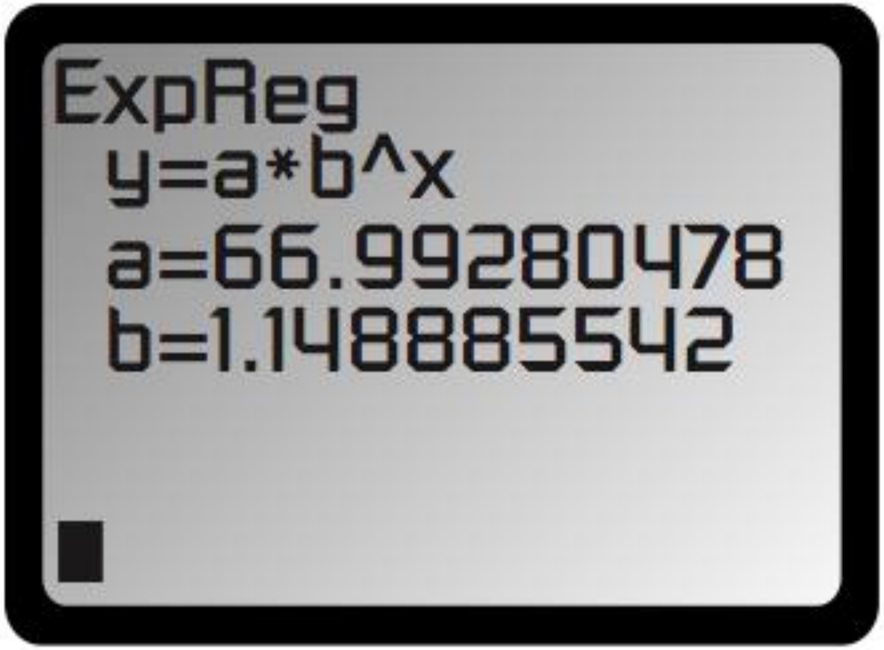
Example Modeling U.S. Population Using Exponential Regression

Make a scatter plot.



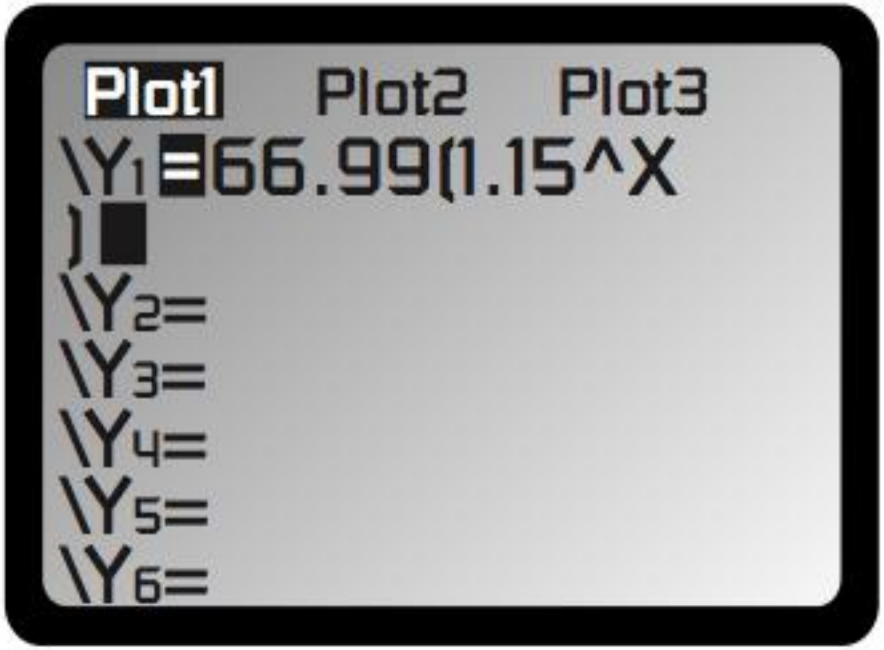
Example **Modeling U.S. Population Using Exponential Regression**

Choose the ExpReg option from your grapher's regression menu, using L_1 for the *x*-values and L_2 as *y*-values.



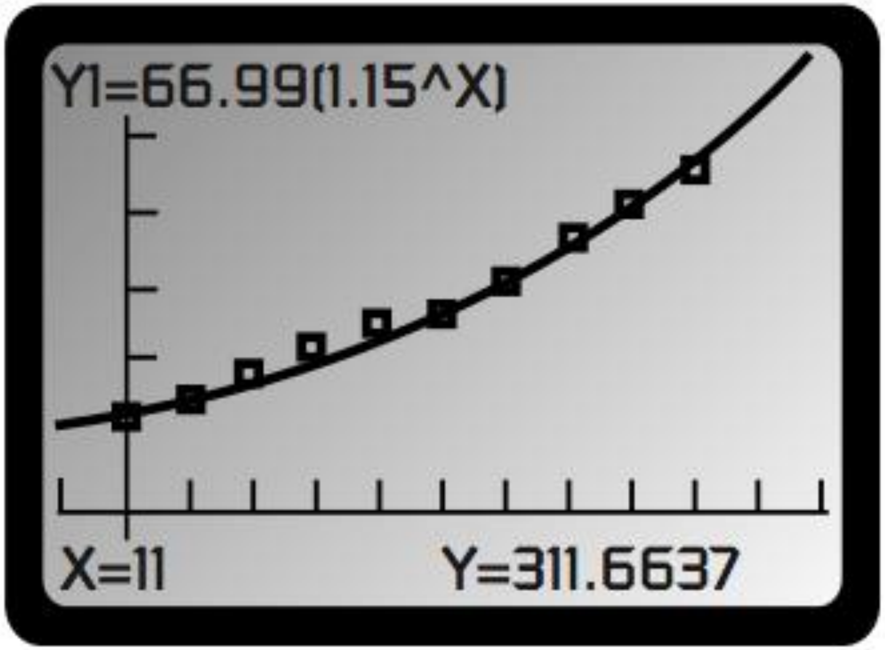
Example Modeling U.S. Population Using Exponential Regression

The regression equation is $f(x) = 66.99(1.15^x)$.



Example Modeling U.S. Population Using Exponential Regression

Graph both the scatter plot and the regression equation. Use the CALC-VALUE option to find $f(11)$.



Example **Modeling U.S. Population Using Exponential Regression**

The year 2000 is represented by $x = 11$, and $f(11) = 311.66$.

Interpret This exponential model estimates the 2000 population to be 311.66 million, an overestimate of approximately 31 million.

Maximum Sustainable Population

Exponential growth is unrestricted, but population growth often is not. For many populations, the growth begins exponentially, but eventually slows and approaches a limit to growth called the **maximum sustainable population**.

Example **Modeling a Rumor**

A high school has 1500 students. 5 students start a rumor, which spreads logistically so that $S(t) = 1500/(1 + 29 \cdot e^{-0.9t})$ models the number of students who have heard the rumor by the end of t days, where $t = 0$ is the day the rumor begins to spread.

(a) How many students have heard the rumor by the end of Day 0?

(b) How long does it take for 1000 students to hear the rumor?

Example **Modeling a Rumor**

5 students start a rumor, $S(t) = 1500 / (1 + 29 \cdot e^{-0.9t})$

where $t = 0$ is the day the rumor begins to spread.

(a) How many students have heard the rumor by the end of Day 0?

$$\text{(a) } S(0) = 1500 / (1 + 29 \cdot e^{-0.9(0)})$$
$$= 1500 / (1 + 29 \cdot 1)$$
$$= 1500 / 30 = 50.$$

So 50 students have heard the rumor by the end of day 0.

Example **Modeling a Rumor**

5 students start a rumor, $S(t) = 1500 / (1 + 29 \cdot e^{-0.9t})$

where $t = 0$ is the day the rumor begins to spread.

(b) How long does it take for 1000 students to hear the rumor?

Solve $1000 = 1500 / (1 + 29 \cdot e^{-0.9t})$ for t.

$t \approx 4.5$. So 1000 students have heard the rumor half way through the fifth day.

Quick Review

Convert the percent to decimal form or the decimal into a percent.

1. 16%

2. 0.05

3. Show how to increase 25 by 8% using a single multiplication.

Solve the equation algebraically.

4. $20 \cdot b^2 = 720$

Solve the equation numerically.

5. $123 \cdot b^3 = 7.872$

Quick Review Solutions

Convert the percent to decimal form or the decimal into a percent.

1. 16% 0.16

2. 0.05 5%

3. Show how to increase 25 by 8% using a single multiplication.

Solve the equation algebraically. $25 \cdot 1.08$

4. $20 \cdot b^2 = 720$ ± 6

Solve the equation numerically.

5. $123 \cdot b^3 = 7.872$ 0.4