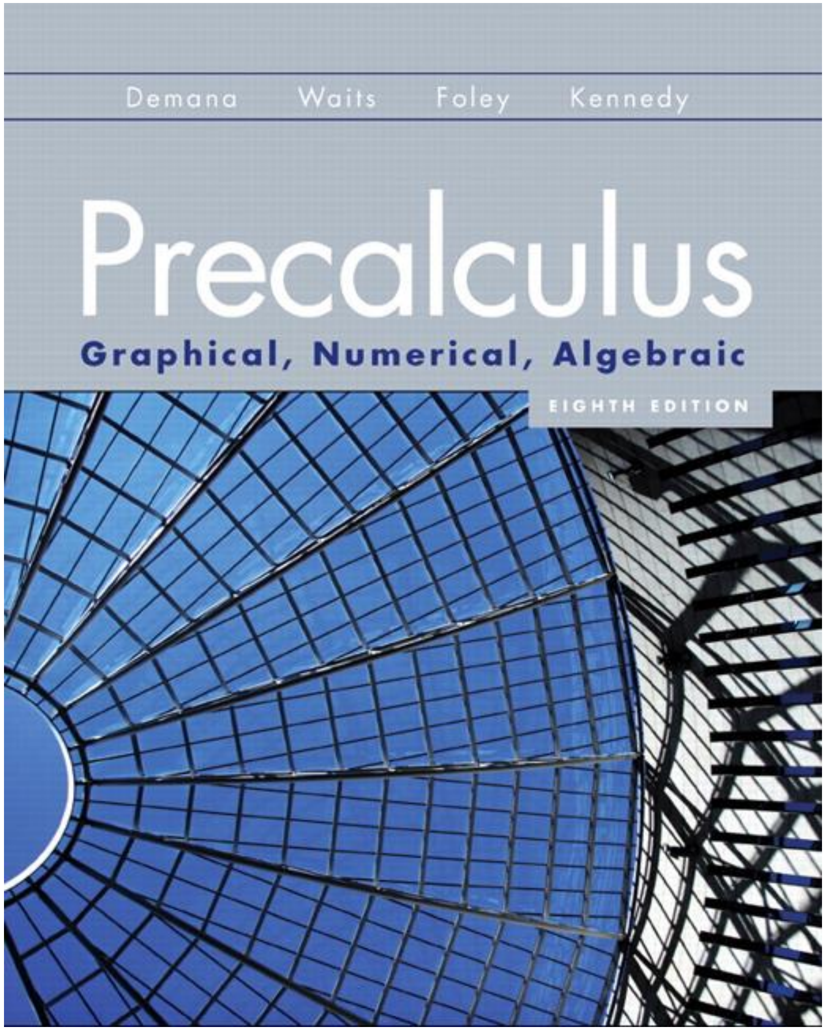
# 3.1

# Exponential and Logistic Functions



# What you'll learn about

- Exponential Functions and Their Graphs
- The Natural Base *e*
- Logistic Functions and Their Graphs
- Population Models

… and why

Exponential and logistic functions model many growth patterns, including the growth of human and animal populations.

# Exponential Functions

Let $a$ and $b$ be real number constants.

An **exponential function** in $x$ is a function that can be written in the form

$$f(x) = a \cdot b^x,$$

where $a$ is nonzero, $b$ is positive, and $b \neq 1$.

The constant $a$ is the *initial value* of $f$ (the value at $x = 0$), and $b$ is the **base**.

# Example Finding an Exponential Function from its Table of Values

Determine formulas for the exponential function $g$ and $h$ whose values are given in the table below.

| $x$ | $g(x)$ | | $h(x)$ | |
|---|---|---|---|---|
| $-2$ | $4/9$ | | $128$ | |
| | | $\times 3$ | | $\times \frac{1}{4}$ |
| $-1$ | $4/3$ | | $32$ | |
| | | $\times 3$ | | $\times \frac{1}{4}$ |
| $0$ | $4$ | | $8$ | |
| | | $\times 3$ | | $\times \frac{1}{4}$ |
| $1$ | $12$ | | $2$ | |
| | | $\times 3$ | | $\times \frac{1}{4}$ |
| $2$ | $36$ | | $1/2$ | |

# Example Finding an Exponential Function from its Table of Values

Determine formulas for the exponential function $g$ and $h$ whose values are given in the table below.

Because $g$ is exponential, $g(x) = a \cdot b^x$. Because $g(0) = 4$, $a = 4$. Because $g(1) = 4 \cdot b^1 = 12$, the base $b = 3$.

So, $g(x) = 4 \cdot 3^x$.

Because $h$ is exponential, $h(x) = a \cdot b^x$. Because $h(0) = 8$, $a = 8$. Because $h(1) = 8 \cdot b^1 = 2$, the base $b = 1/4$.

So, $h(x) = 8 \cdot \left(\frac{1}{4}\right)^x$.

# Exponential Growth and Decay

For any exponential function $f(x) = a \cdot b^x$ and any real number $x$,

$$f(x+1) = b \cdot f(x).$$

If $a > 0$ and $b > 1$, the function $f$ is increasing and is an **exponential growth function**.

The base $b$ is its **growth factor**.

If $a > 0$ and $b < 1$, the function $f$ is decreasing and is an **exponential decay function**.

The base $b$ is its **decay factor**.

# Example Transforming Exponential Functions

Describe how to transform the graph of $f(x) = 2^x$ into the graph of each function. Graph each function.

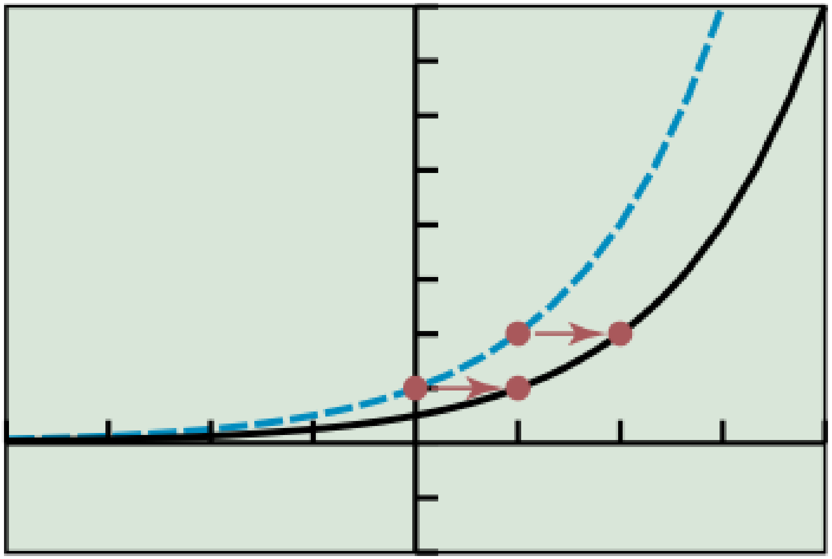
a. $g(x) = 2^{x-1}$ b. $h(x) = 2^{-x}$ c. $k(x) = 3 \cdot 2^x$

# Example **Transforming Exponential Functions**

Describe how to transform the graph of $f(x) = 2^x$ into the graph of each function. Graph each function.

a. $g(x) = 2^{x-1}$

The graph of $g(x) = 2^{x-1}$ is obtained by translating the graph of $f(x) = 2^x$ by 1 unit to the right.



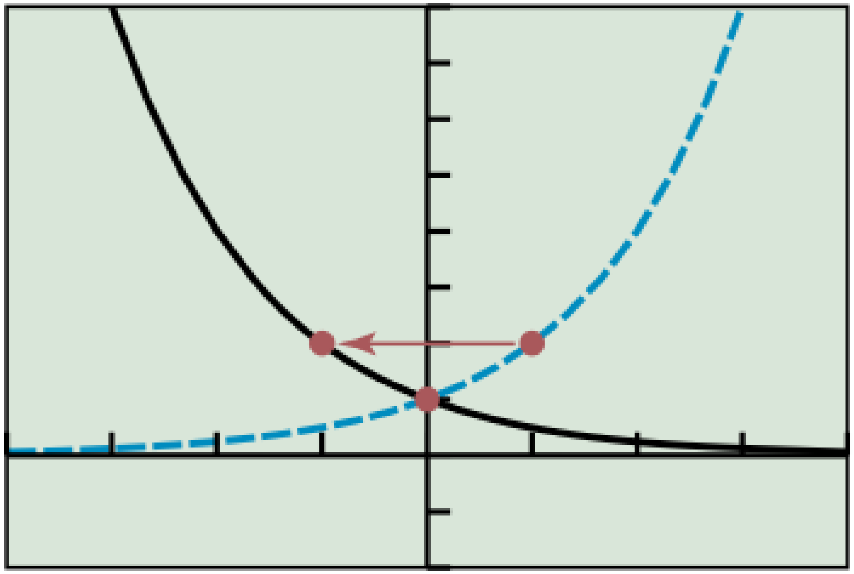
[–4, 4] by [–2, 8]

# Example Transforming Exponential Functions

Describe how to transform the graph of $f(x) = 2^x$ into the graph of each function. Graph each function.

b. $h(x) = 2^{-x}$

The graph of $h(x) = -e$ is obtained by reflecting the graph of $f(x) = e^x$ across the $x$-axis.



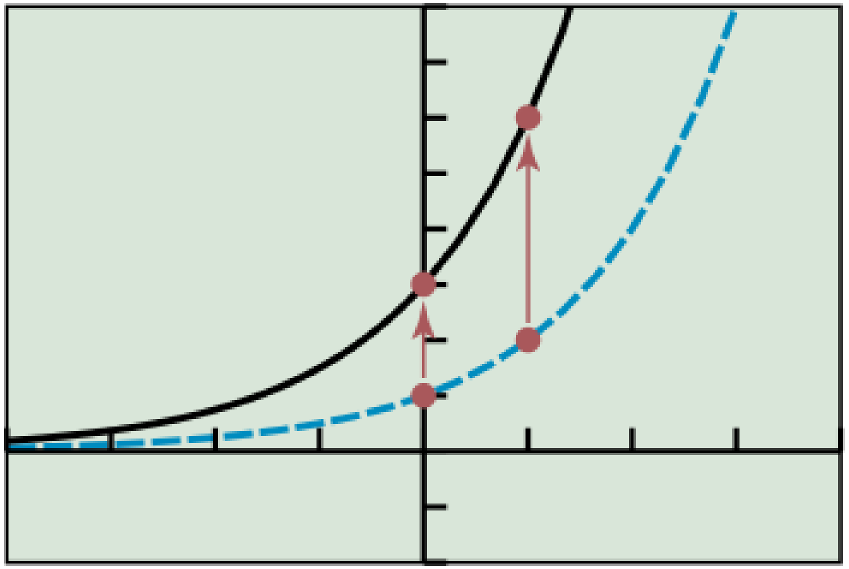
[–4, 4] by [–2, 8]

# Example Transforming Exponential Functions

Describe how to transform the graph of $f(x) = 2^x$ into the graph of each function. Graph each function.

c. $k(x) = 3 \cdot 2^x$

The graph of $k(x) = 3 \cdot 2^x$ is obtained by vertically stretching the graph of $f(x) = 2^x$ by a factor of 3.



[–4, 4] by [–2, 8]

# The Natural Base $e$

$$e = \lim_{x \to \infty} \left(1 + \frac{1}{x}\right)^x$$

# Exponential Functions and the Base $e$

Any exponential function $f(x) = a \cdot b^x$ can be rewritten as

$$f(x) = a \cdot e^{kx},$$

for any appropriately chosen real number constant $k$.

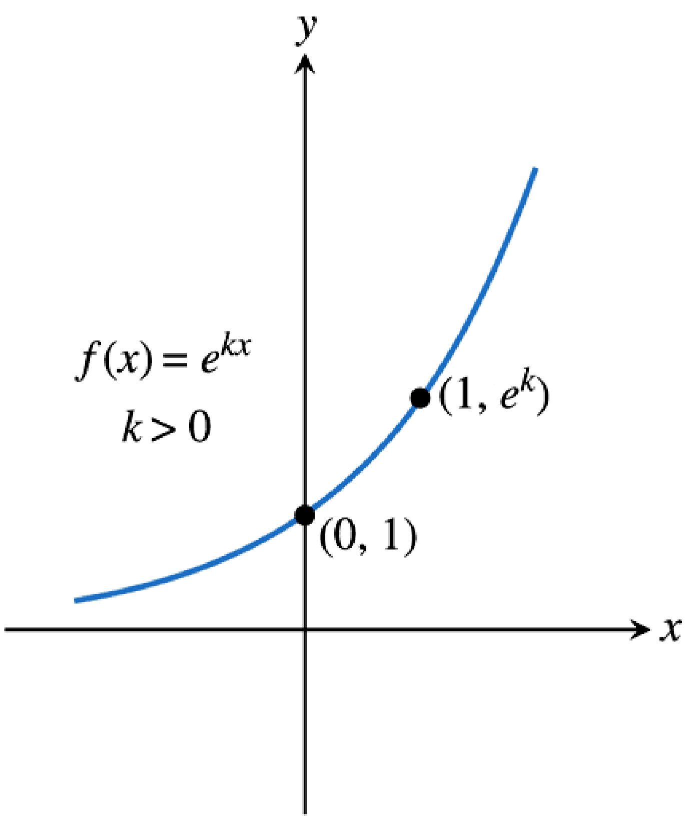
If $a > 0$ and $k > 0$, $f(x) = a \cdot e^{kx}$

is an exponential growth function.

If $a > 0$ and $k < 0$, $f(x) = a \cdot e^{kx}$

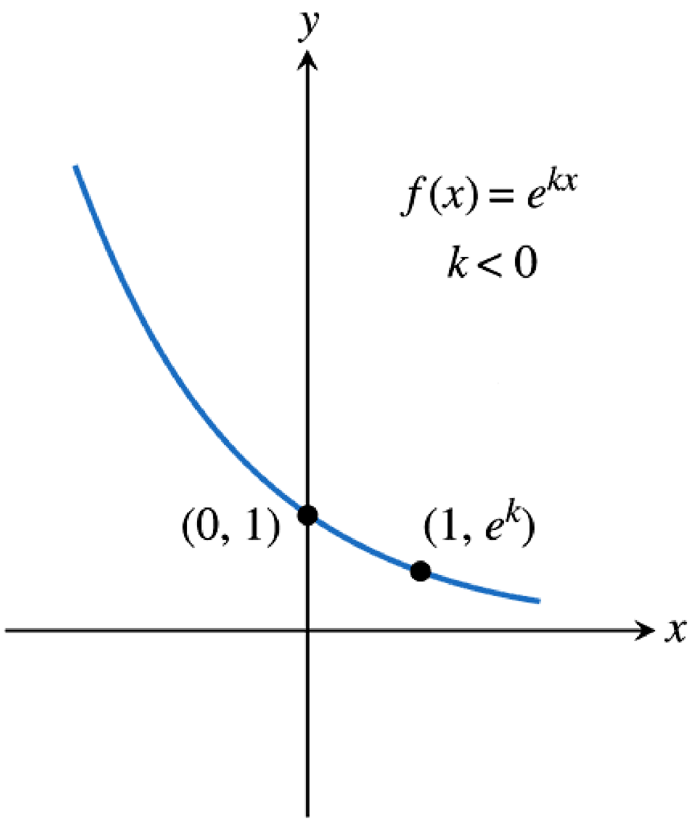
is an exponential decay function.

# Exponential Functions and the Base $e$




(a)

**Exponential growth function**




(b)

**Exponential decay function**

# Example **Transforming Exponential Functions**

Describe how to transform the graph of $f(x) = e^x$ into the graph of each function. Graph each function.

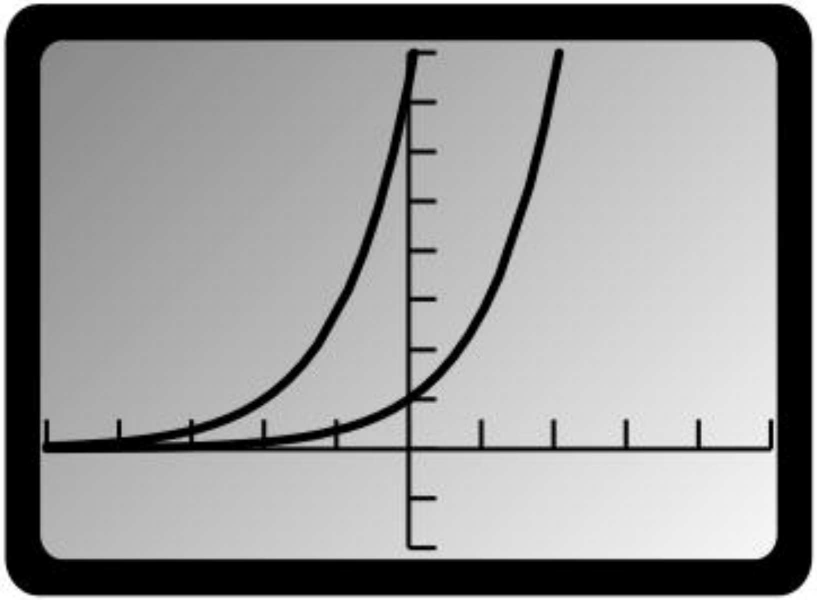
a. $g(x) = e^{x+2}$ b. $h(x) = -e^x$ c. $k(x) = e^x - 1$

# Example Transforming Exponential Functions

Describe how to transform the graph of $f(x) = e^x$ into the graph of each function. Graph each function.

a. $g(x) = e^{x+2}$

The graph of $g(x) = e^{x+2}$ is obtained by translating the graph of $f(x) = e^x$ by 2 units to the left.



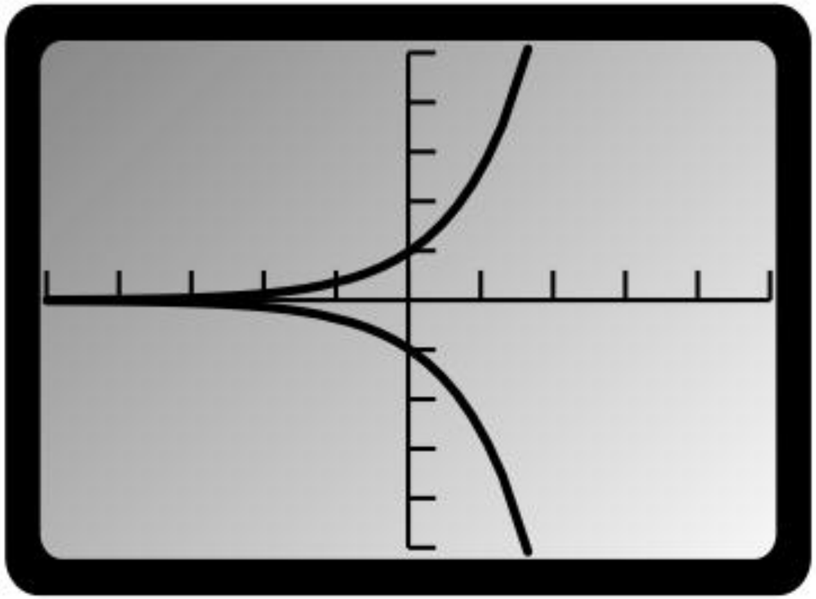
$[-5, 5]$ by $[-2, 8]$

# Example **Transforming Exponential Functions**

Describe how to transform the graph of $f(x)=e^x$ into the graph of each function. Graph each function.

b. $h(x)=-e^x$

The graph of $h(x)=-e$ is obtained by reflecting the graph of $f(x)=e^x$ across the $x$-axis.



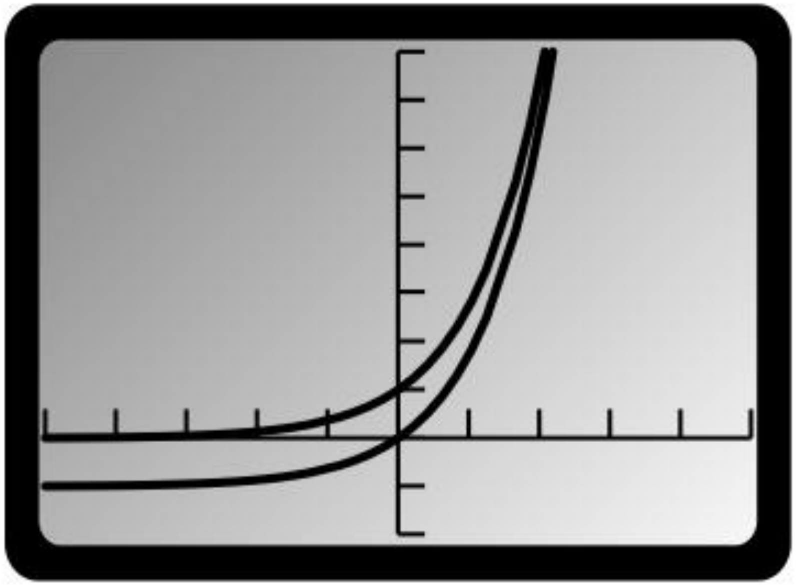
[−5, 5] by [−5, 5]

# Example Transforming Exponential Functions

Describe how to transform the graph of $f(x) = e^x$ into the graph of each function. Graph each function.

c. $k(x) = e^x - 1$

The graph of $k(x) = e^x - 1$ is obtained by translating the graph of $f(x) = e^x$ by 1 unit down.



$[-5, 5]$ by $[-2, 8]$

# Logistic Growth Functions

Let $a$, $b$, $c$, and $k$ be positive constants, with $b < 1$. A **logistic growth function** in $x$ is a function that can be written in the form

$$f(x) = \frac{c}{1 + a \cdot b^x} \quad \text{or} \quad f(x) = \frac{c}{1 + a \cdot e^{-kx}}$$

where the constant $c$ is the **limit to growth.**

# Quick Review

Evaluate the expression without using a calculator.

1. $\sqrt[3]{-125}$

2. $\sqrt[3]{\dfrac{27}{64}}$

3. $27^{4/3}$

Rewrite the expression using a single positive exponent.

4. $\left(a^{-3}\right)^2$

Use a calculator to evaluate the expression.

5. $\sqrt[5]{3.71293}$

# Quick Review Solutions

Evaluate the expression without using a calculator.

1. $\sqrt[3]{-125}$ $-5$

2. $\sqrt[3]{\dfrac{27}{64}}$ $\dfrac{3}{4}$

3. $27^{4/3}$ $81$

Rewrite the expression using a single positive exponent.

4. $\left(a^{-3}\right)^{2}$ $\dfrac{1}{a^{6}}$

Use a calculator to evaluate the expression.

5. $\sqrt[5]{3.71293}$ $1.3$