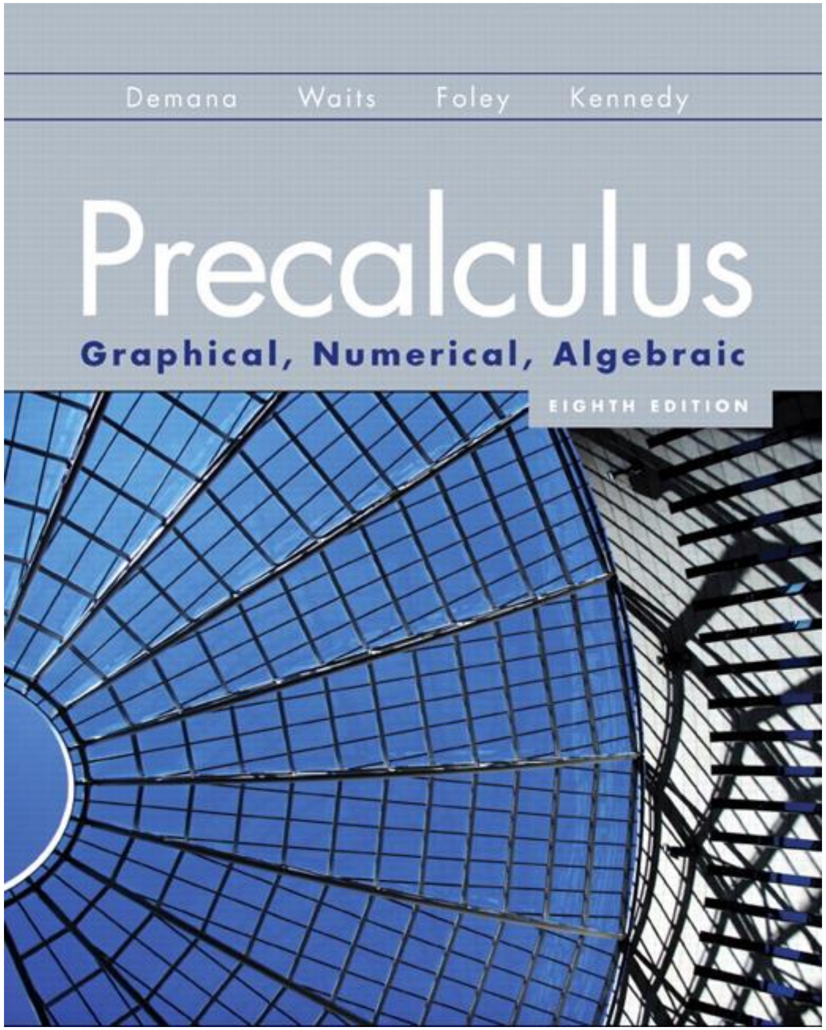
2.6

Graphs of Rational Functions



What you'll learn about

- Rational Functions
- Transformations of the Reciprocal Function
- Limits and Asymptotes
- Analyzing Graphs of Rational Functions

… and why

Rational functions are used in calculus and in scientific applications such as inverse proportions.

Rational Functions

Let f and g be polynomial functions with $g(x) \neq 0$. Then the function given by

$$r(x) = \frac{f(x)}{g(x)}$$

is a **rational function**.

Example **Finding the Domain of a Rational Function**

Find the domain of f and use limits to describe the behavior at value(s) of x not in its domain.

$$f(x) = \frac{2}{x+2}$$

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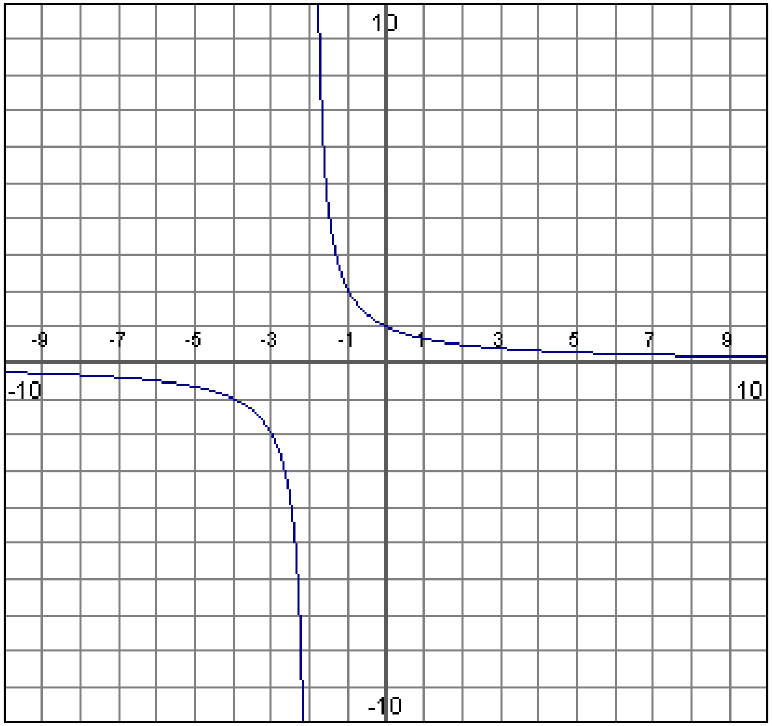
Find the domain of f and use limits to describe the behavior at value(s) of x not in its domain.

$$f(x) = \frac{2}{x+2}$$

The domain of f is all real numbers $x \neq -2$. Use a graph of the function to find

$$\lim_{x \to -2^+} f(x) = \infty \text{ and}$$

$$\lim_{x \to -2^-} f(x) = -\infty.$$



The Reciprocal Function

$$f(x) = \frac{1}{x}$$

Domain: $(-\infty, 0) \cup (0, \infty)$

Range: $(-\infty, 0) \cup (0, \infty)$

Continuity: All $x \neq 0$

Decreasing on $(-\infty, 0) \cup (0, \infty)$

Symmetric with respect to origin (an odd function)

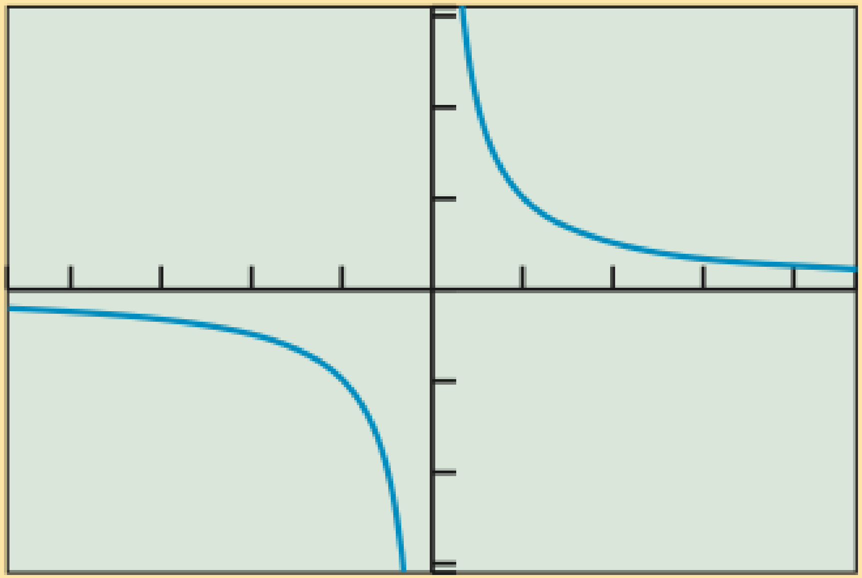
Unbounded

No local extrema

Horizontal asymptote: $y = 0$

Vertical asymptote: $x = 0$

End behavior: $\lim_{x \to -\infty} f(x) = \lim_{x \to \infty} f(x) = 0$



[–4.7, 4.7] by [–3.1, 3.1]

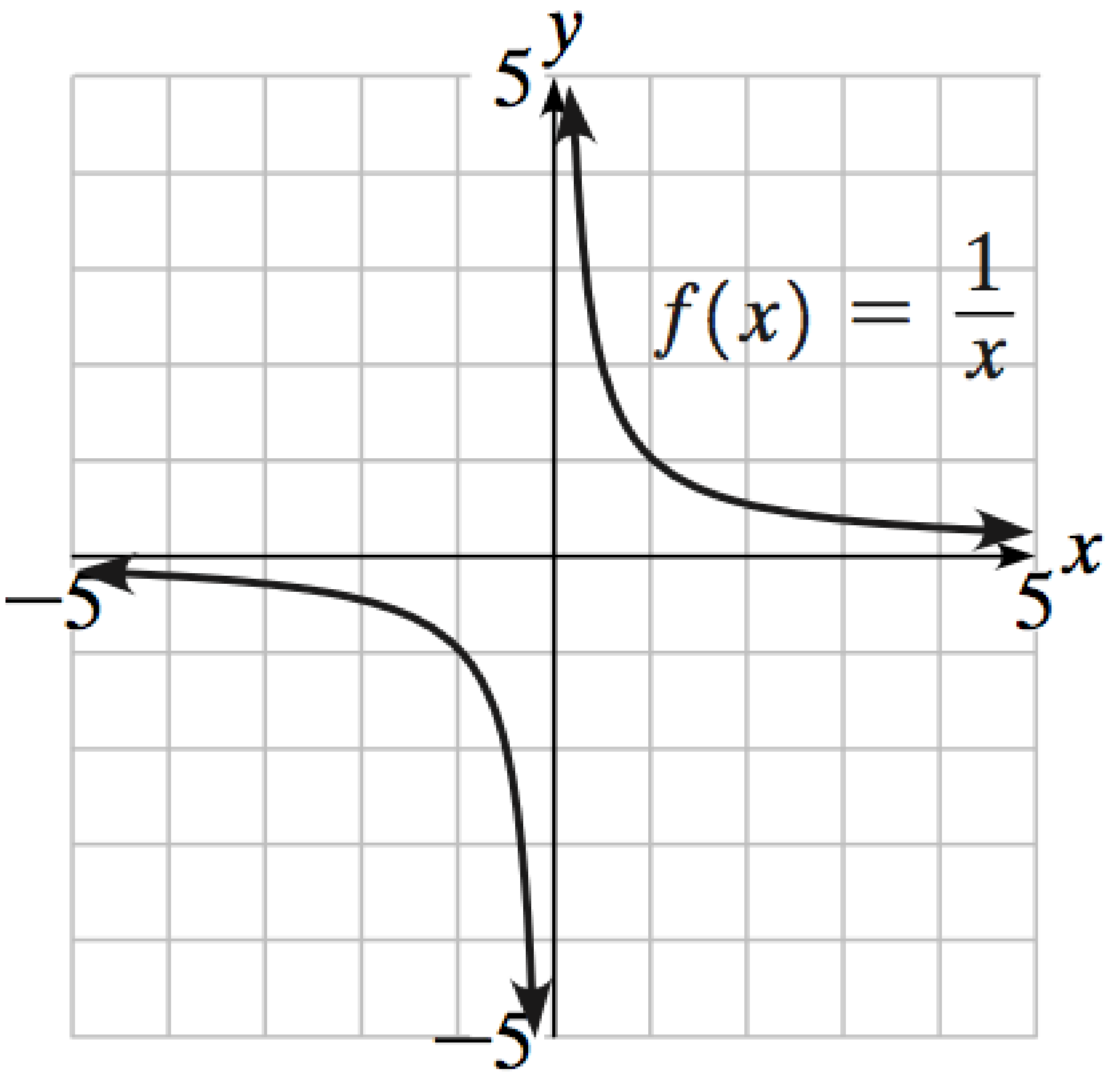
Example **Transforming the Reciprocal Function**

Describe how the graph of the function can be obtained by tranforming the graph of $f(x) = \dfrac{1}{x}$. Identify the horizontal and vertical asymptotes and use limits to describe the corresponding behavior. Sketch the graph of the function.

a. $g(x) = -\dfrac{2}{x}$ b. $h(x) = -\dfrac{2}{x+3}$ c. $k(x) = \dfrac{2x+4}{x+3}$

The graph of $f(x) = \dfrac{1}{x}$ is shown.

Example **Transforming the Reciprocal Function**



Example Transforming the Reciprocal Function

a. $g(x) = -\frac{2}{x} = -2f(x)$

reflect f across x-axis

stretch vertically by factor of 2

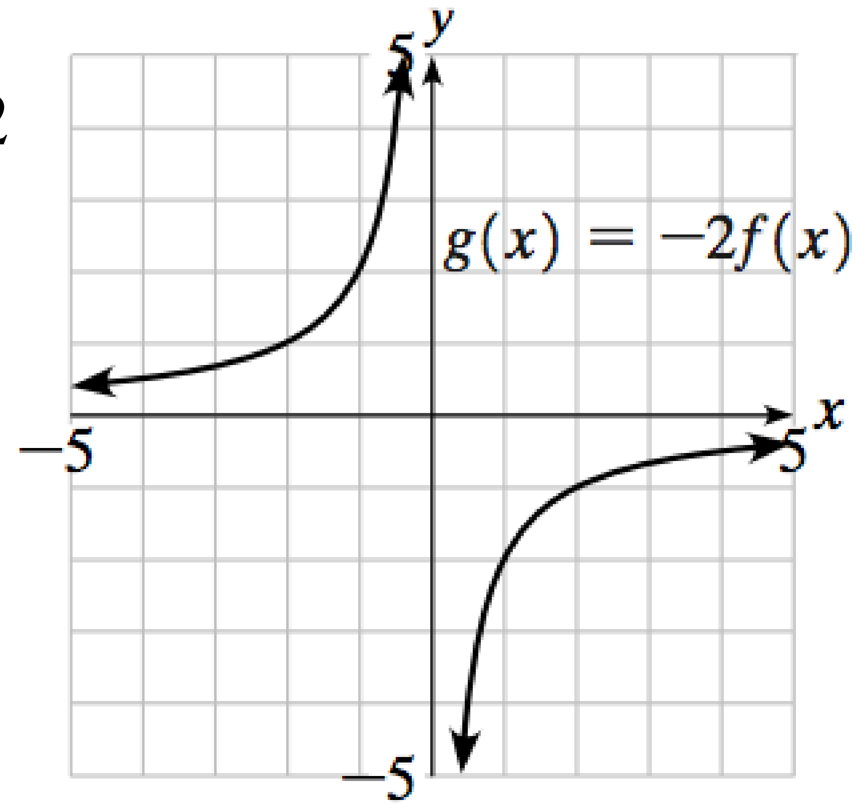
vertical asymptote $x = 0$

horizontal asymptote $y = 0$

$\lim_{x\to\infty} g(x) = \lim_{x\to-\infty} g(x) = 0$

$\lim_{x\to 0^+} g(x) = -\infty$ and

$\lim_{x\to 0^-} g(x) = \infty.$



Example Transforming the Reciprocal Function

b. $h(x) = -\dfrac{2}{x+3} = g(x+3) - 2f(x+3)$

reflect f across x-axis

stretch vertically by factor of 2

translate 3 units left

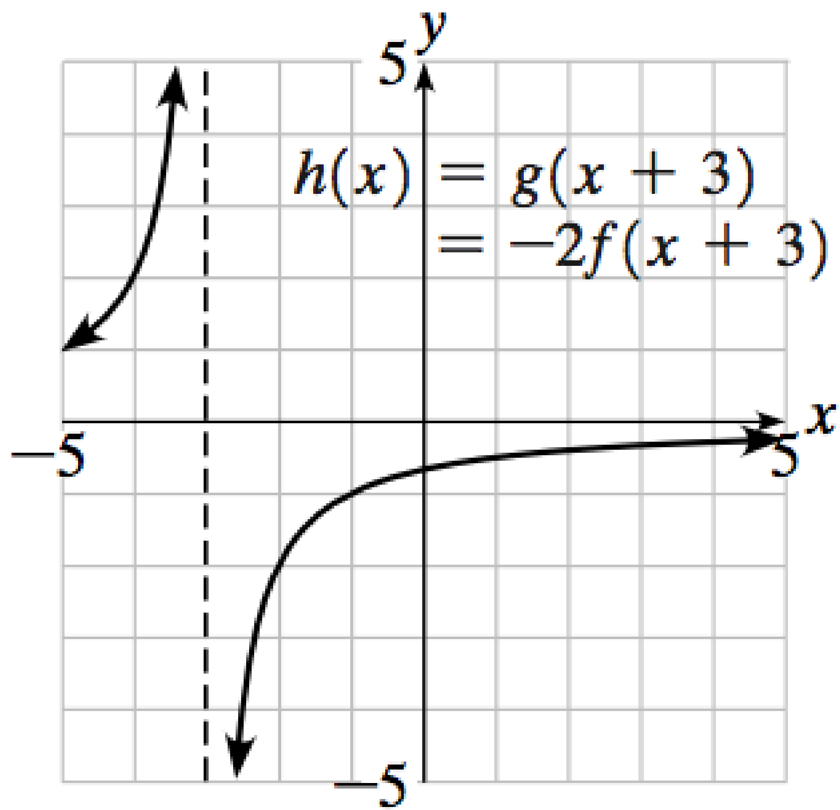
vertical asymptote $x = -3$

horizontal asymptote $y = 0$

$\lim_{x\to\infty} h(x) = \lim_{x\to-\infty} h(x) = 0$

$\lim_{x\to-3^+} h(x) = -\infty$ and

$\lim_{x\to-3^-} h(x) = \infty.$



Example **Transforming the Reciprocal Function**

c. $k(x) = \dfrac{2x+4}{x+3} = 2 - \dfrac{2}{x+3} = h(2) + 2$

$$= g(x+3) + 2 = -2f(x+3) + 2$$

reflect f across x-axis

stretch vertically by factor of 2

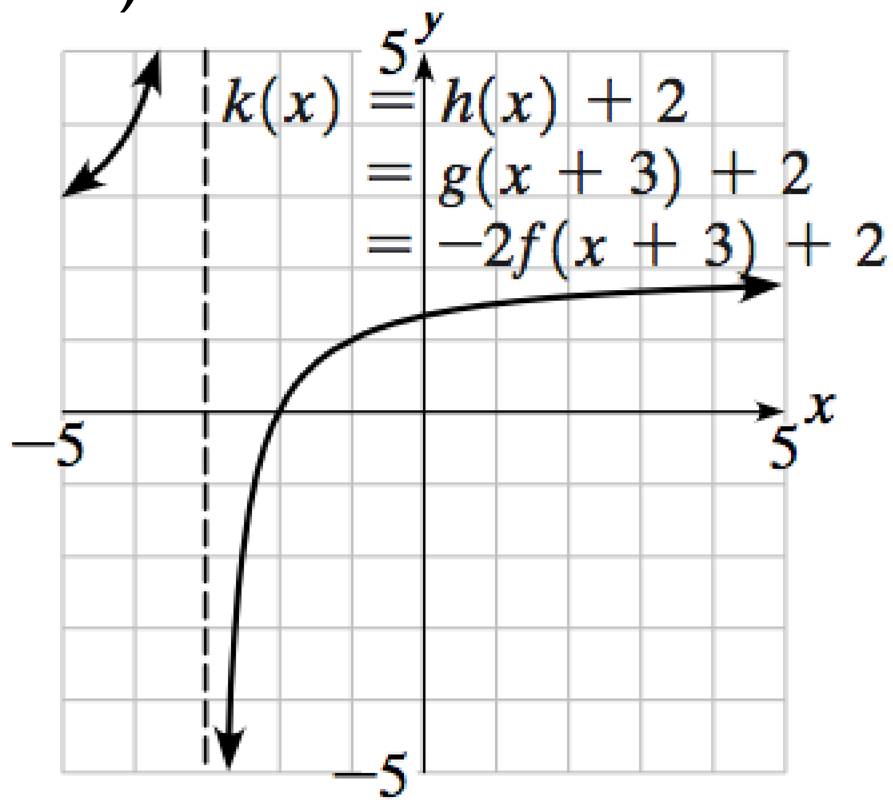
translate 3 units left, 2 upward

vertical asymptote $x = -3$

horizontal asymptote $y = 2$

$\lim\limits_{x\to\infty} h(x) = \lim\limits_{x\to-\infty} h(x) = 2$

$\lim\limits_{x\to-3^+} h(x) = -\infty, \quad \lim\limits_{x\to-3^-} h(x) = \infty.$



Graph of a Rational Function

The graph of $y = f(x)/g(x) = (a_n x^n + \ldots)/(b_m x^m + \ldots)$ has the following characteristics:

1. End behavior asymptote :

If $n < m$, the end behavior asymptote is the horizontal asymptote $y = 0$.

If $n = m$, the end behavior asymptote is the horizontal asymptote $y = a_n / b_m$.

If $n > m$, the end behavior asymptote is the quotient polynomial function $y = q(x)$, where $f(x) = g(x)q(x) + r(x)$. There is no horizontal asymptote.

Graph of a Rational Function

The graph of $y = f(x)/g(x) = (a_n x^n + \ldots)/(b_m x^m + \ldots)$ has the following characteristics:

2. Vertical asymptotes: These occur at the zeros of the denominator, provided that the zeros are not also zeros of the numerator of equal or greater multiplicity.

3. x **- intercepts:** These occur at the zeros of the numerator, which are not also zeros of the denominator.

4. y **- intercepts:** This is the value of $f(0)$, if defined.

Example **Finding Asymptotes of Rational Functions**

Find the asymptotes of the function $f(x) = \dfrac{2(x+3)(x-3)}{(x+1)(x+5)}$.

Example Finding Asymptotes of Rational Functions

Find the asymptotes of the function $f(x)=\dfrac{2(x+3)(x-3)}{(x+1)(x+5)}$.

There are vertical asymptotes
at the zeros of the denominator:
$x=-1$ and $x=-5$.

The end behavior asymptote is at $y=2$.

Example **Graphing a Rational Function**

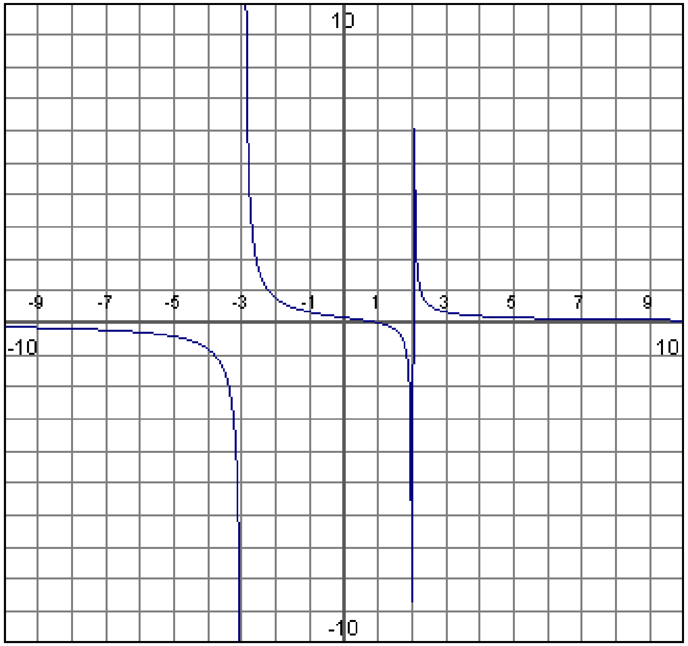
Find the asymptotes and intercepts of

$$f(x)=\frac{x-1}{(x-2)(x+3)} \text{ and graph } f(x).$$

Example Graphing a Rational Function

$$f(x) = \frac{x-1}{(x-2)(x+3)}$$

Numerator is zero when $x = 1$ so the x-intercept is 1. $f(0) = 1/6$, the y-intercept is 1/6. Denominator is zero when $x = 2$ and $x = -3$, vertical asymptotes at $x = 2$ and $x = -3$. Degree of numerator is less than the degree of denominator, horizontal asymptote at $y = 0$.



Quick Review

Use factoring to find the real zeros of the function.

1. $f(x) = 2x^2 + 7x + 6$

2. $f(x) = x^2 - 16$

3. $f(x) = x^2 + 16$

4. $f(x) = x^3 - 27$

Find the quotient and remainder when $f(x)$ is divided by $d(x)$.

5. $f(x) = 5x - 3, \quad d(x) = x$

Quick Review Solutions

Use factoring to find the real zeros of the function.

1. $f(x) = 2x^2 + 7x + 6$ $\quad x = -3/2,\ x = -2$

2. $f(x) = x^2 - 16$ $\quad x = \pm 4$

3. $f(x) = x^2 + 16$ $\quad$ no real zeros

4. $f(x) = x^3 - 27$ $\quad x = 3$

Find the quotient and remainder when $f(x)$ is divided by $d(x)$.

5. $f(x) = 5x - 3,\ \ d(x) = x$ $\quad 5; -3$