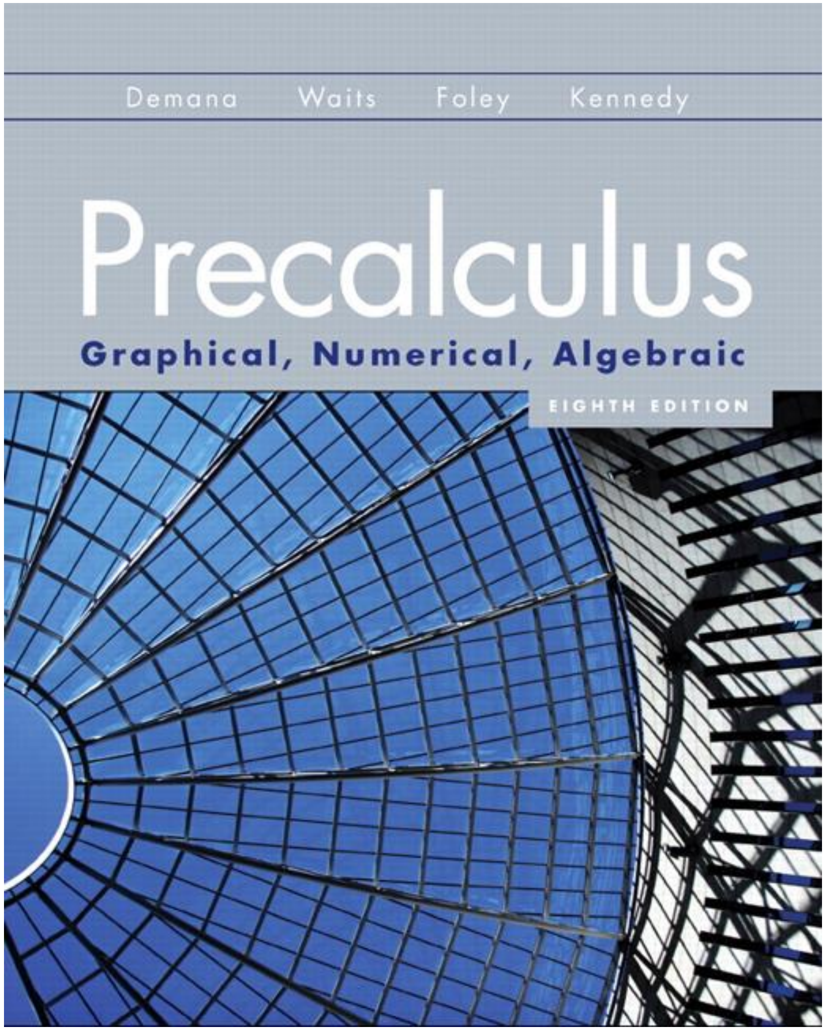
2.5
Complex Zeros and the Fundamental Theorem of Algebra



What you'll learn about

- Two Major Theorems
- Complex Conjugate Zeros
- Factoring with Real Number Coefficients

… and why

These topics provide the complete story about the zeros and factors of polynomials with real number coefficients.

Fundamental Theorem of Algebra

A polynomial function of degree n has n complex zeros (real and nonreal). Some of these zeros may be repeated.

Linear Factorization Theorem

If $f(x)$ is a polynomial function of degree $n > 0$, then $f(x)$ has precisely n linear factors and

$$f(x) = a(x - z_1)(x - z_2)...(x - z_n)$$

where a is the leading coefficient of $f(x)$ and $z_1, z_2, ..., z_n$ are the complex zeros of $f(x)$. The z_i are not necessarily distinct numbers; some may be repeated.

Fundamental Polynomial Connections in the Complex Case

The following statements about a polynomial function f are equivalent if k is a complex number:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$
2. k is a zero of the function f.
3. $x - k$ is a factor of $f(x)$.

Example **Exploring Fundamental Polynomial Connections**

Write the polynomial function in standard form, identify the zeros of the function and the *x*-intercepts of its graph.

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The function $f(x) = (x - 3i)(x + 3i) = x^2 + 9$ has two zeros: $x = 3i$ and $x = -3i$. Because the zeros are not real, the graph of f has no x-intercepts.

Complex Conjugate Zeros

Suppose that $f(x)$ is a polynomial function with real coefficients. If a and b are real numbers with $b \neq 0$ and $a + bi$ is a zero of $f(x)$, then its complex conjugate $a - bi$ is also a zero of $f(x)$.

Example **Finding a Polynomial from Given Zeros**

Write a polynomial function of minimum degree in standard form with real coefficients whose zeros include -3, $3 - i$, and $2 + 5i$.

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Because the coefficients are real and $3 - i$ is a zero, $3 + i$ must also be a zero. Therefore, $x - (3 - i)$ and $x - (3 + i)$ must be factors.

Similarly, $2 - 5i$ must be a zero since $2 - 5i$ is a zero. So $x - (2 - 5i)$ and $x - (2 + 5i)$ must both be factors of f.

Because -3 is a real zeros, $x + 3$ must be a factor.

Example Finding a Polynomial from Given Zeros

$$f(x) = (x+3)[x-(3-i)][x-(3+i)][x-(2-5i)][x-(2+5i)]$$

$$= (x+3)[(x-3)+i][(x-3)-i][(x-2)+5i][(x-2)-5i]$$

$$(x-3)^2 - i^2 \qquad (x-2)^2 - (5i)^2$$

$$(x^2-6x+9)-(-1) \qquad (x^2-4x+4)-(-25)$$

$$x^2-6x+10 \qquad x^2-4x+29$$

$$f(x) = (x+3)(x^2-6x+10)(x^2-4x+29)$$

$$= (x^3-3x^2-8x+10)(x^2-4x+29)$$

$$= x^5-7x^4+33x^3-25x^2-352x+870$$

Factors of a Polynomial with Real Coefficients

Every polynomial function with real coefficients can be written as a product of linear factors and irreducible quadratic factors, each with real coefficients.

Example Factoring a Polynomial

Write $f(x) = 3x^5 + x^4 - 24x^3 - 8x^2 - 27x - 9$ as a product of linear and irreducible quadratic factors, each with real coefficients.

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The Rational Zeros Theorem provides the candidates for the rational zeros of f. The graph of f suggests which candidates to try first. Using synthetic division, find that $x = -1/3$ is a zero. Thus,

$$f(x) = 3x^5 + x^4 - 24x^3 - 8x^2 - 27x - 9$$

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$$f(x) = 3x^5 + x^4 - 24x^3 - 8x^2 - 27x - 9$$
$$= \left(x + \frac{1}{3}\right)(3)\left(x^4 - 8x^2 - 9\right)$$
$$= 3\left(x + \frac{1}{3}\right)\left(x^2 - 9\right)\left(x^2 + 1\right)$$
$$= 3\left(x + \frac{1}{3}\right)(x - 3)(x + 3)\left(x^2 + 1\right)$$

Quick Review

Perform the indicated operation, and write the result in the form $a+bi$.

1. $(2+3i)+(-1+5i)$
2. $(-3+2i)(3-4i)$

Factor the quadratic equation.

3. $2x^2-9x-5$

Solve the quadratic equation.

4. $x^2-6x+10=0$

List all potential rational zeros.

5. $4x^4-3x^2+x+2$

Quick Review Solutions

Perform the indicated operation, and write the result in the form $a + bi$.

1. $(2+3i)+(-1+5i)$ $\quad 1+8i$
2. $(-3+2i)(3-4i)$ $\quad -1+18i$

Factor the quadratic equation.

3. $2x^2 - 9x - 5$ $\quad (2x+1)(x-5)$

Solve the quadratic equation.

4. $x^2 - 6x + 10 = 0$ $\quad x = 3 \pm i$

List all potential rational zeros.

5. $4x^4 - 3x^2 + x + 2$ $\quad \pm 2, \pm 1, \pm 1/2, \pm 1/4$