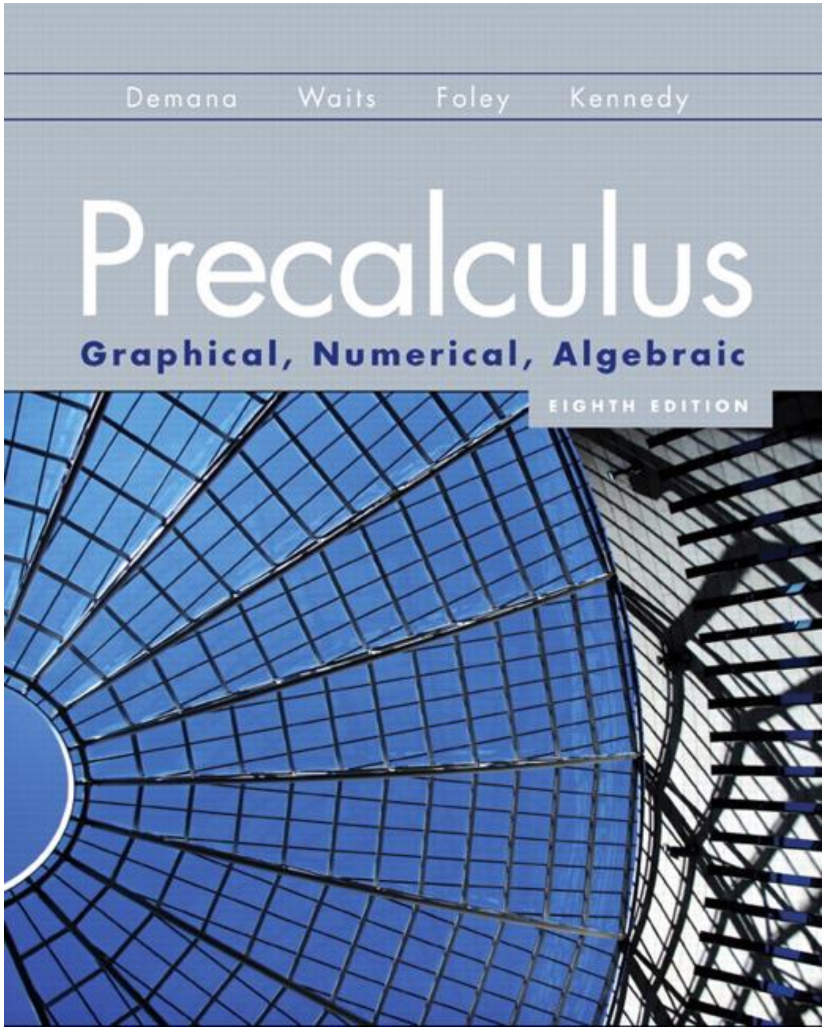
2.4

Real Zeros of Polynomial Functions



PEARSON
Addison Wesley

What you'll learn about

- Long Division and the Division Algorithm
- Remainder and Factor Theorems
- Synthetic Division
- Rational Zeros Theorem
- Upper and Lower Bounds

… and why

These topics help identify and locate the real zeros of polynomial functions.

Division Algorithm for Polynomials

Let $f(x)$ and $d(x)$ be polynomials with the degree of f greater than or equal to the degree of d, and $d(x) \neq 0$. Then there are unique polynomials $q(x)$ and $r(x)$, called the **quotient** and **remainder**, such that

$$f(x) = d(x) \cdot q(x) + r(x)$$

where either $r(x) = 0$ or the degree of r is less than the degree of d.

The function $f(x)$ in the division algorithm is the **dividend**, and $d(x)$ is the **divisor**.

If $r(x) = 0$, we say $d(x)$ **divides evenly** into $f(x)$.

Example Using Polynomial Long Division

Use long division to find the quotient and remainder when $2x^4 + x^3 - 3$ is divided by $x^2 + x + 1$.

Example Using Polynomial Long Division

$$\begin{array}{r} 2x^2 - x - 1 \\ x^2 + x + 1 \overline{\smash{)}\, 2x^4 + x^3 + 0x^2 + 0x - 3} \\ \underline{2x^4 + 2x^3 + 2x^2 } \\ -x^3 - 2x^2 + 0x - 3 \\ \underline{-x^3 - x^2 - x } \\ -x^2 + x - 3 \\ \underline{-x^2 - x - 1} \\ 2x - 2 \end{array}$$

$$\left(2x^4 + x^3 - 3\right) \div \left(x^2 + x + 1\right) = 2x^2 - x - 1 + \frac{2x - 2}{x^2 + x + 1}$$

Remainder Theorem

If polynomial $f(x)$ is divided by $x - k$, then the remainder is $r = f(k)$.

Example Using the Remainder Theorem

Find the remainder when $f(x) = 2x^2 - x + 12$ is divided by $x + 3$.

Example Using the Remainder Theorem

Find the remainder when $f(x) = 2x^2 - x + 12$ is divided by $x + 3$.

$$r = f(-3) = 2(-3)^2 - (-3) + 12 = 33$$

Factor Theorem

A polynomial function $f(x)$ has a factor $x-k$ if and only if $f(k)=0$.

Fundamental Connections for Polynomial Functions

For a polynomial function f and a real number k the following statements are equivalent:

1. $x = k$ is a solution (or root) of the equation $f(x) = 0$
2. k is a zero of the function f.
3. k is an x-intercept of the graph of $y = f(x)$.
4. $x - k$ is a factor of $f(x)$.

Example Using Synthetic Division

Divide $3x^3 - 2x^2 + x - 5$ by $x - 1$ using synthetic division.

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1⌋	3	−2	1	−5
	3			

1⌋	3	−2	1	−5
		3	1	2
	3	1	2	−3

$$\frac{3x^3 - 2x^2 + x - 5}{(x-1)} = 3x^2 + x + 2 - \frac{3}{(x-1)}$$

Rational Zeros Theorem

Suppose f is a polynomial function of degree $n \geq 1$ of the form

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_0,$$

with every coefficient an integer and $a_0 \neq 0$. If $x = p/q$ is a rational zero of f, where p and q have no common integer factors other than 1, then

- p is an integer factor of the constant coefficient a_0,
- q is an integer factor of the leading coefficient a_n.

Upper and Lower Bound Tests for Real Zeros

Let f be a polynomial function of degree $n \geq 1$ with a positive leading coefficient. Suppose $f(x)$ is divided by $x - k$ using synthetic division.

- If $k \geq 0$ and every number in the last line is nonnegative (positive or zero), then k is an *upper bound* for the real zeros of f.
- If $k \leq 0$ and the numbers in the last line are alternately nonnegative and nonpositive, then k is a *lower bound* for the real zeros of f.

Example **Finding the Real Zeros of a Polynomial Function**

Find all of the real zeros of

$$f(x) = 2x^5 - x^4 - 2x^3 - 14x^2 - 6x + 36$$

and identify them as rational or irrational.

Example **Finding the Real Zeros of a Polynomial Function**

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$$f(x) = 2x^5 - x^4 - 2x^3 - 14x^2 - 6x + 36$$

and identify them as rational or irrational.

$$\textit{Potential Rational Zeros}:$$

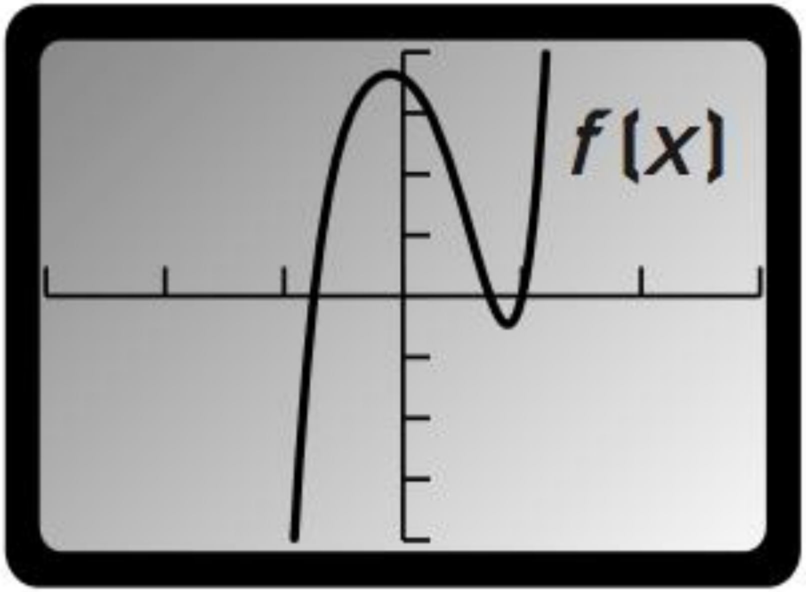
$$\frac{\text{Factors of } 36}{\text{Factors of } 2}: \pm 1, \pm 2, \pm 3, \pm 4, \pm 6, \pm 9, \pm 12, \pm 18, \pm 36,$$

$$\pm\frac{1}{2}, \pm\frac{3}{2}, \pm\frac{9}{2}$$

Example **Finding the Real Zeros of a Polynomial Function**

The graph suggests that only $x = -\frac{3}{2}$, $x = \frac{3}{2}$, and $x = 2$ be considered. Synthetic division shows that only $x = -\frac{3}{2}$, and $x = 2$ are zeros of f, and

$$f(x) = (2x+3)(x-2)(x^3+2x-6).$$



[–6, 6] by [–40, 40]

Example **Finding the Real Zeros of a Polynomial Function**

Now let $g(x) = x^3 + 2x - 6.$

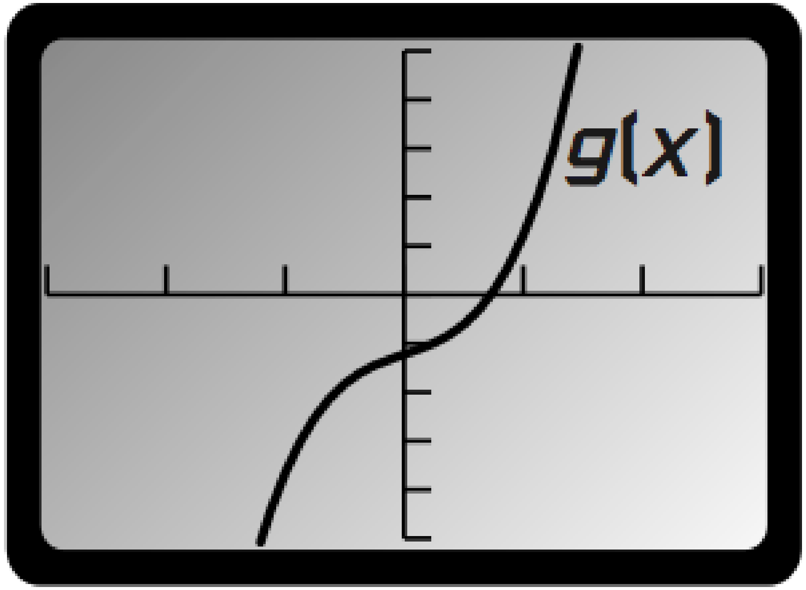
potential zeros :

$\pm 1, \pm 2, \pm 3, \pm 6$

but the graph shows that

these values are not zeros of g.

So f has no more rational zeros.

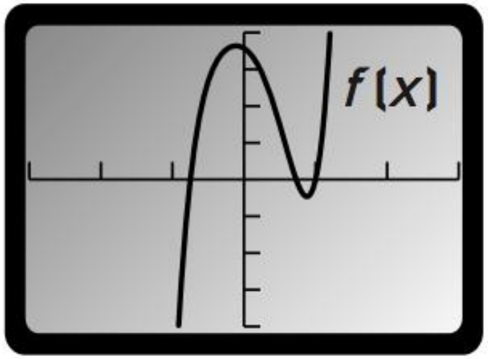


$[-6, 6]$ by $[-25, 25]$

Example Finding the Real Zeros of a Polynomial Function

Using grapher methods with either function $(f \text{ or } g)$ shows that an irrational zero exists at $x \approx 1.456$. The upper and lower bound tests can be applied to f or g to show that there are no zeros outside the viewing windows shown.

The real zeros of f are the rational numbers $x = -\frac{3}{2}$ and $x = 2,$ and an irrational number $x \approx 1.456.$



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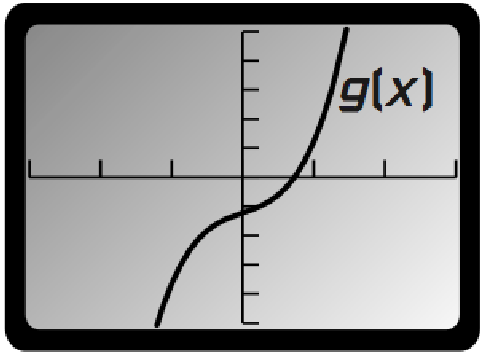

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Example **Finding the Real Zeros of a Polynomial Function**

Find all of the real zeros of $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8$.

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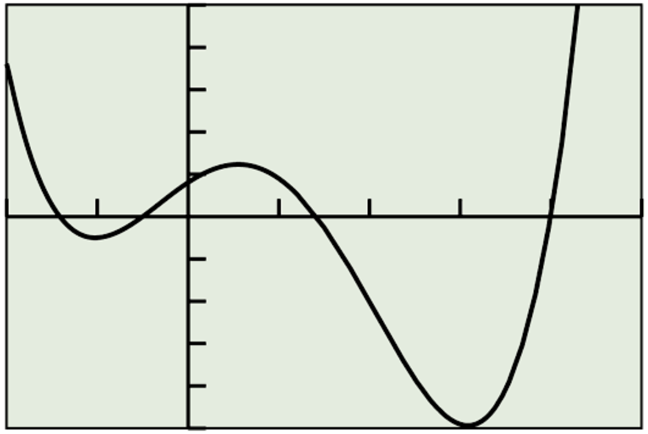
$$\textit{Potential Rational Zeros}:$$

$$\frac{\text{Factors of } 8}{\text{Factors of } 2} : \frac{\pm 1, \pm 2, \pm 4, \pm 8}{\pm 1, \pm 2} = \pm 1, \pm 2, \pm 4, \pm 8, \pm \frac{1}{2}$$

Compare the x-intercepts of the graph and the list of possibilities, and decide that 4 and $-1/2$ are potential rational zeros.



[–2, 5] by [–50, 50]

Example **Finding the Real Zeros of a Polynomial Function**

Find all of the real zeros of $f(x) = 2x^4 - 7x^3 - 8x^2 + 14x + 8.$

$$\begin{array}{r|rrrrr} 4 & 2 & -7 & -8 & 14 & 8 \\ & & 8 & 4 & -16 & -8 \\ \hline & 2 & 1 & -4 & -2 & 0 \end{array}$$

This tells us that

$$2x^4 - 7x^3 - 8x^2 + 14x + 8 = (x-4)(2x^3 + x^2 - 4x - 2).$$

Example **Finding the Real Zeros of a Polynomial Function**

Find all of the real zeros of $f(x)=2x^4-7x^3-8x^2+14x+8.$

$$\begin{array}{r|rrrr} -1/2 & 2 & 1 & -4 & -2 \\ & & -1 & 0 & 2 \\ \hline & 2 & 0 & -4 & 0 \end{array}$$

This tells us that

$$2x^4-7x^3-8x^2+14x+8=2(x-4)\left(x+\frac{1}{2}\right)\left(x^2-2\right).$$

The real zeros are 4, $-\frac{1}{2}, \pm\sqrt{2}.$

Quick Review

Rewrite the expression as a polynomial in standard form.

1. $\dfrac{2x^3 + 3x^2 + x}{x}$

2. $\dfrac{2x^5 - 8x^3 + x^2}{2x^2}$

Factor the polynomial into linear factors.

3. $x^3 - 16x$

4. $x^3 + x^2 - 4x - 4$

5. $6x^2 - 24$

Quick Review Solutions

Rewrite the expression as a polynomial in standard form.

1. $\dfrac{2x^3 + 3x^2 + x}{x}$ $\quad 2x^2 + 3x + 1$

2. $\dfrac{2x^5 - 8x^3 + x^2}{2x^2}$ $\quad x^3 - 4x + \dfrac{1}{2}$

Factor the polynomial into linear factors.

3. $x^3 - 16x$ $\quad x(x+4)(x-4)$

4. $x^3 + x^2 - 4x - 4$ $\quad (x+1)(x+2)(x-2)$

5. $6x^2 - 24$ $\quad 6(x+2)(x-2)$