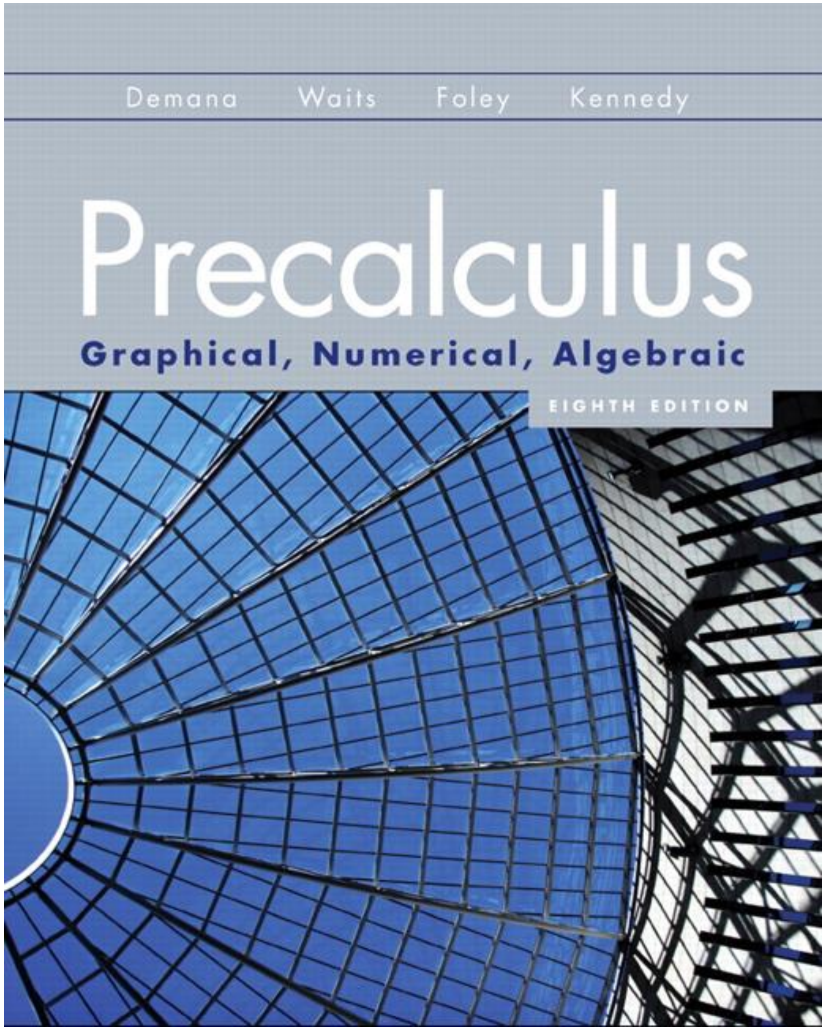
1.7

Modeling with Functions



What you'll learn about

- Functions from Formulas
- Functions from Graphs
- Functions from Verbal Descriptions
- Functions from Data

… and why

Using a function to model a variable under observation in terms of another variable often allows one to make predictions in practical situations, such as predicting the future growth of a business based on data.

Example A Maximum Value Problem

A square of side x inches is cut out of each corner of an 8 in. by 15 in. piece of cardboard and the sides are folded up to form an open-topped box.

(a) Write the volume V as a function of x.

(b) Find the domain of V as a function of x.

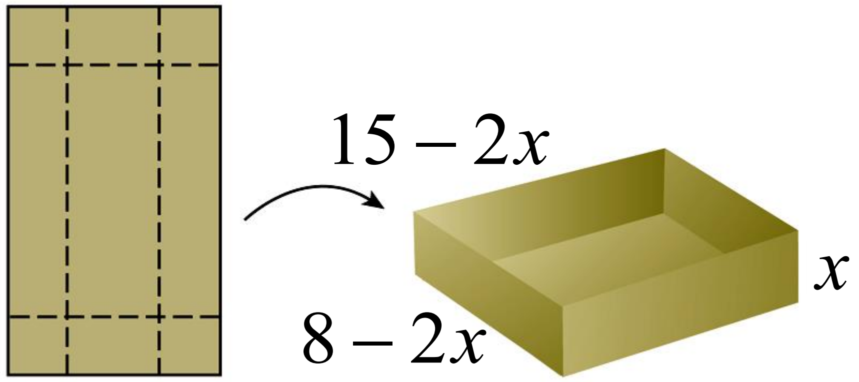
(c) Graph V as a function of x over the domain found in part (b) and use the maximum finder on your grapher to determine the maximum volume such a box can hold.

(d) How big should the cut-out squares be in order to produce the box of maximum volume?

Solution

A square of side x inches is cut out of each corner of an 8 in. by 15 in. piece of cardboard and the sides are folded up to form an open-topped box.

(a) Write the volume V as a function of x.



$$V = x(8-2x)(15-2x)$$

Solution

A square of side x inches is cut out of each corner of an 8 in. by 15 in. piece of cardboard and the sides are folded up to form an open-topped box.

(b) Find the domain of V as a function of x.

$$V = x(8-2x)(15-2x)$$

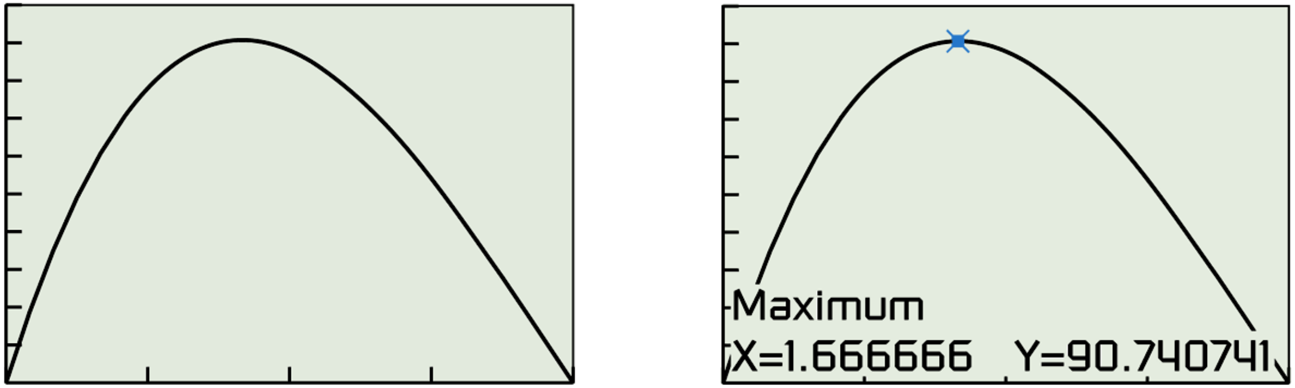
The depth of x must be nonnegative, as must the side length and width.

The domain is [0,4] where the endpoints give a box with no volume.

Solution

$$V = x(8-2x)(15-2x)$$

(c) Graph V as a function of x over the domain found in part (b) and use the maximum finder on your grapher to determine the maximum volume such a box can hold.



[0, 4] by [0, 100]
(a)

[0, 4] by [0, 100]
(b)

The maximum occurs at the point (5/3, 90.74).

The maximum volume is about 90.74 in.^{3}.

Solution

A square of side x inches is cut out of each corner of an 8 in. by 15 in. piece of cardboard and the sides are folded up to form an open-topped box.

(d) How big should the cut-out squares be in order to produce the box of maximum volume?

Each square should have sides of one-and-two-thirds inches.

Example **Finding the Model and Solving**

Grain is leaking through a hole in a storage bin at a constant rate of 5 cubic inches per minute. The grain forms a cone-shaped pile on the ground below. As it grows, the height of the cone always remains equal to its radius. If the cone is one foot tall now, how tall will it be in one hour?

Solution

The volume of a cone is $V = \left(1/3\right)\pi r^2 h$. Since the height always equals the radius, $V = \left(1/3\right)\pi h^3$. When $h = 12$ inches, the volume will be $V = (1/3)\pi\left(12\right) = 576\pi$ in.^{3}. One hour later, the volume will have grown by (60 min)(5 in.^{3} / min) = 300 in^3. The total volume will be $\left(576\pi + 300\right)$ in^3.

$$\left(1/3\right)\pi h^3 = 576\pi + 300$$

$$h^3 = \frac{3\left(576\pi + 300\right)}{\pi}$$

$$h = \sqrt[3]{\frac{3\left(576\pi + 300\right)}{\pi}} \approx 12.63 \text{ inches}$$

Constructing a Function from Data

Given a set of data points of the form (x, y), to construct a formula that approximates y as a function of x:

1. Make a scatter plot of the data points. The points do not need to pass the vertical line test.
2. Determine from the shape of the plot whether the points seem to follow the graph of a familiar type of function (line, parabola, cubic, sine curve, etc.).
3. Transform a basic function of that type to fit the points as closely as possible.

Example Curve-Fitting with Technology

The table shows that the number of patent applications in the United States increased from 1993 to 2003. Find both a linear and a quadratic regression model for this data. Which appears to be the better model of the data?

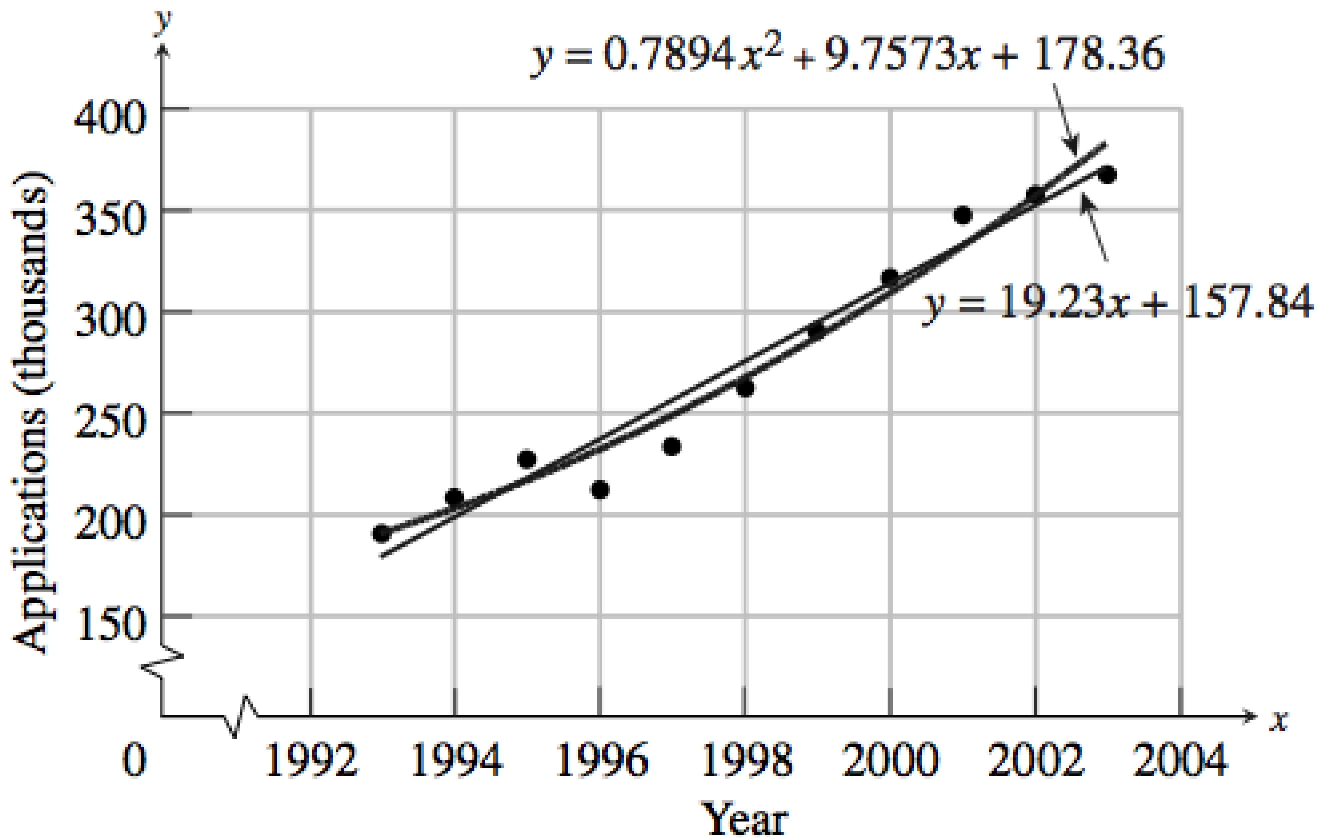
U.S. Patent Applications

Year	Applications (thousands)
1993	189.4
1994	206.9
1995	226.6
1996	211.6
1997	233.0
1998	261.5
1999	289.5
2000	315.8
2001	346.6
2002	357.5
2003	367.0

Source: U.S. Bureau of the Census, *Statistical Abstract of the United States*, 2003 (Washington, D.C., 2003).

Solution

Use a grapher to compute the linear and quadratic regression, using $x = 0$ for 1993, $x = 1$ for 1994, …



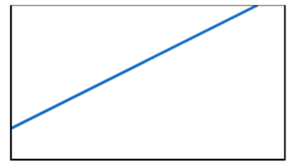
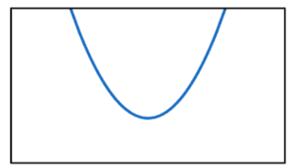
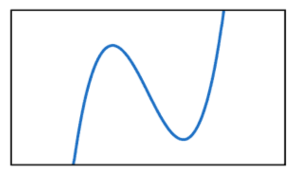
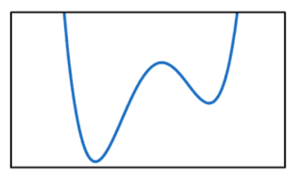
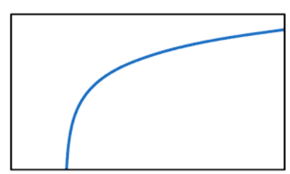
The linear regression model is $y = 19.23x + 1\ 57.84$.

The quadratic regression model is

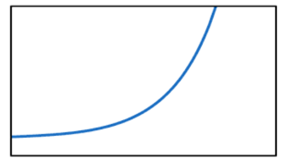
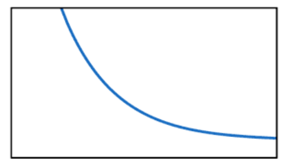
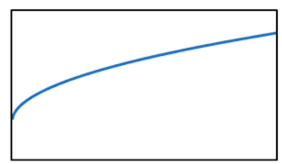
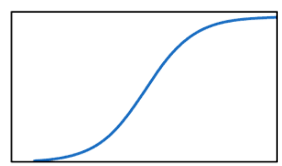
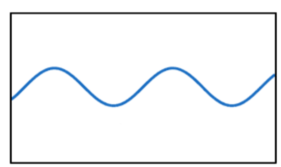
$y = 0.7894x^2 + 9.7573x + 178.36$.

The quadratic regression equation appears to model the data better than the linear regression equation.

Functions

Regression Type	Equation	Graph	Applications
Linear (Chaper 2)	$y = ax + b$		Fixed cost plus variable cost, linear growth, free-fall velocity, simple interest, linear depreciation, many others
Quadratic (Chapter 2)	$y = ax^2 + bx + c$ (requires at least 3 points)		Position during free fall, projectile motion, parabolic reflectors, area as a function of linear dimension, quadratic growth, etc.
Cubic (Chapter 2)	$y = ax^3 + bx^2 + cx + d$ (requires at least 4 points)		Volume as a function of linear dimension, cubic growth, miscellaneous applications where quadratic regression does not give a good fit
Quartic (Chapter 2)	$y = ax^4 + bx^3 + cx^2 + dx + e$ (requires at least 5 points)		Quartic growth, miscellaneous applications where quadratic and cubic regression do not give a good fit
Natural logarithmic (ln) (Chapter 3)	$y = a + b \ln x$ (requires $x > 0$)		Logarithmic growth, decibels (sound), Richter scale (earthquakes), inverse exponential models

Functions (cont'd)

Regression Type	Equation	Graph	Applications
Exponential $(b > 1)$ (Chapter 3)	$y = a \cdot b^x$		Exponential growth, compound interest, population models
Exponential $(0 < b < 1)$ (Chapter 3)	$y = a \cdot b^x$		Exponential decay, depreciation, temperature loss of a cooling body, etc.
Power (requires $x, y > 0$) (Chapter 2)	$y = a \cdot x^b$		Inverse-square laws, Kepler's third law
Logistic (Chapter 3)	$y = \dfrac{c}{1 + a \cdot e^{-bx}}$		Logistic growth: spread of a rumor, population models
Sinusoidal (Chapter 4)	$y = a \sin(bx + c) + d$		Periodic behavior: harmonic motion, waves, circular motion, etc.

Quick Review

Solve the given formula for the given variable.

1. **Area of a Triangle** Solve for b: $A = \frac{1}{2}bh$
2. **Volume of a Right Circular Cylinder**

   Solve for h: $V = \frac{1}{3}\pi r^2 h$
3. **Volume of a Sphere** Solve for r: $V = \frac{4}{3}\pi r^3$
4. **Surface Area of a Sphere** Solve for r: $A = 4\pi r^2$
5. **Simple Interest** Solve for P: $I = Prt$

Quick Review Solutions

Solve the given formula for the given variable.

1. Solve for b: $A = \frac{1}{2}bh$ $\qquad b = \frac{2A}{h}$

2. Solve for h: $V = \frac{1}{3}\pi r^2 h$ $\qquad h = \frac{3V}{\pi r^2}$

3. Solve for r: $V = \frac{4}{3}\pi r^3$ $\qquad r = \sqrt[3]{\frac{3V}{4\pi}}$

4. Solve for r: $A = 4\pi r^2$ $\qquad r = \sqrt{\frac{A}{4\pi}}$

5. Solve for P: $I = Prt$ $\qquad P = \frac{I}{rt}$

Chapter Test

Find the (a) domain and (b) range of the function.

1. $h(x) = (x-2)^2 + 5$

2. $k(x) = \dfrac{1}{\sqrt{9-x^2}}$

3. Is the following function continuous at $x = 0$?

$$f(x) = \begin{cases} 2x + 3 \text{ if } x > 0 \\ 3 - x^2 \text{ if } x \leq 0 \end{cases}$$

4. Find all (a) vertical asymptotes and (b) horizontal asymptotes of the function $y = \dfrac{3x}{x-4}$.

Chapter Test

5. State the interval(s) on which $y = \frac{x^3}{6}$ is increasing.
6. Tell whether the function is bounded above, bounded below or bounded. $g(x) = \frac{6x}{x^2 + 1}$
7. Use a grapher to find all (a) relative maximum values and (b) relative minimum values. $y = x^3 - 3x$
8. State whether the function is even, odd, or neither. $y = 3x^2 - 4|x|$

Chapter Test

9. Find a formula for f^{-1}. $f(x)=\dfrac{6}{x+4}$

10. Find an expression for $(f \circ g)(x)$ given $f(x)=\sqrt{x}$ and $g(x)=x^2-4$.

Chapter Test Solutions

Find the (a) domain and (b) range of the function.

1. $h(x) = (x-2)^2 + 5$ (a) $(-\infty, \infty)$ (b) $[5, \infty)$

2. $k(x) = \dfrac{1}{\sqrt{9-x^2}}$ (a) $(-3, 3)$ (b) $[1/3, \infty)$

3. Is the following function continuous at $x = 0$?

$$f(x) = \begin{cases} 2x+3 \text{ if } x > 0 \\ 3 - x^2 \text{ if } x \le 0 \end{cases}$$

yes

4. Find all (a) vertical asymptotes and (b) horizontal asymptotes of the function $y = \dfrac{3x}{x-4}$. (a) $x = 4$ (b) $y = 3$

Chapter Test Solutions

5. State the interval(s) on which $y = \dfrac{x^3}{6}$ is increasing. $(-\infty, \infty)$

6. Tell whether the function is bounded above, bounded below or bounded. $g(x) = \dfrac{6x}{x^2 + 1}$ bounded

7. Use a grapher to find all (a) relative maximum values and (b) relative minimum values. $y = x^3 - 3x$

   $(a)\ 2 \qquad (b)\ -2$

8. State whether the function is even, odd, or neither.

   $y = 3x^2 - 4|x|$ even

Chapter Test

9. Find a formula for f^{-1}. $f(x)=\dfrac{6}{x+4}$

$f^{-1}(x) = 6/x - 4$

10. Find an expression for $(f \circ g)(x)$ given

$f(x)=\sqrt{x}$ and $g(x)=x^2-4$.

$\sqrt{x^2-4}$