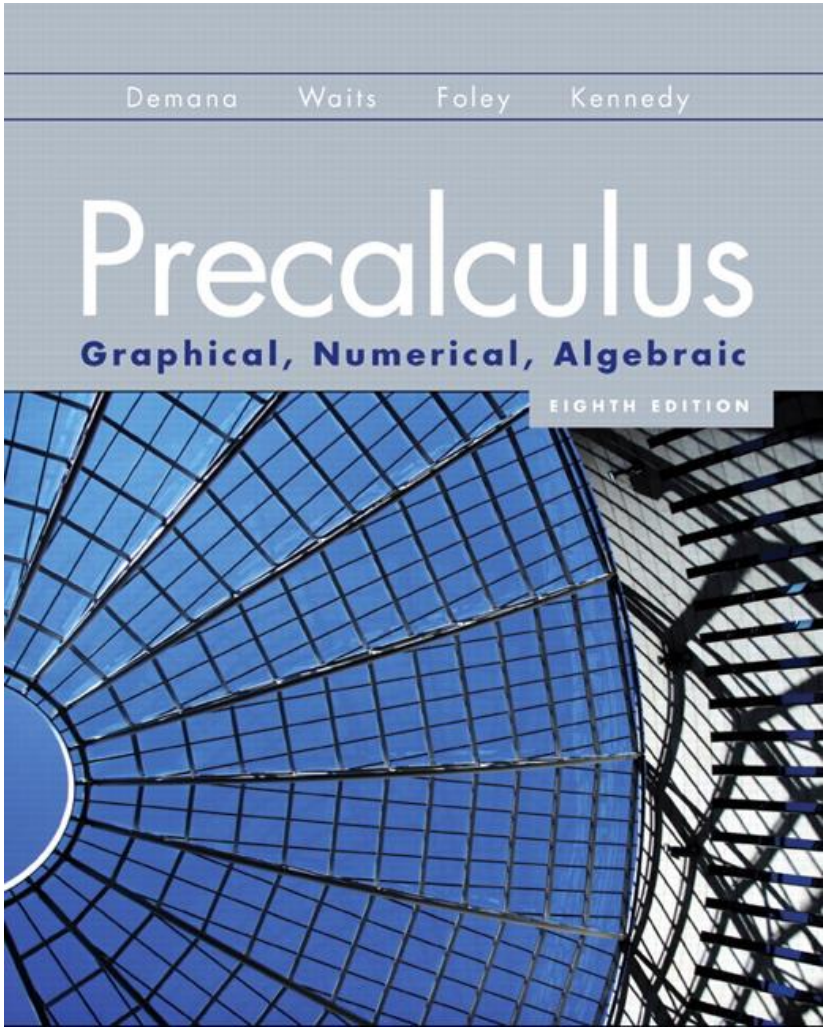
1.6

Graphical Transformations



What you'll learn about

- Transformations
- Vertical and Horizontal Translations
- Reflections Across Axes
- Vertical and Horizontal Stretches and Shrinks
- Combining Transformations

… and why

Studying transformations will help you to understand the relationships between graphs that have similarities but are not the same.

Transformations

In this section we relate graphs using **transformations**, which are functions that map real numbers to real numbers.

Rigid transformations, which leave the size and shape of a graph unchanged, include horizontal translations, vertical translations, reflections, or any combination of these.

Nonrigid transformations, which generally distort the shape of a graph, include horizontal or vertical stretches and shrinks.

Vertical and Horizontal Translations

Let c be a positive real number. Then the following transformations result in translations of the graph of $y = f(x)$.

Horizontal Translations

$y = f(x - c)$	a translation to the right by c units
$y = f(x + c)$	a translation to the left by c units

Vertical Translations

$y = f(x) + c$	a translation up by c units
$y = f(x) - c$	a translation down by c units

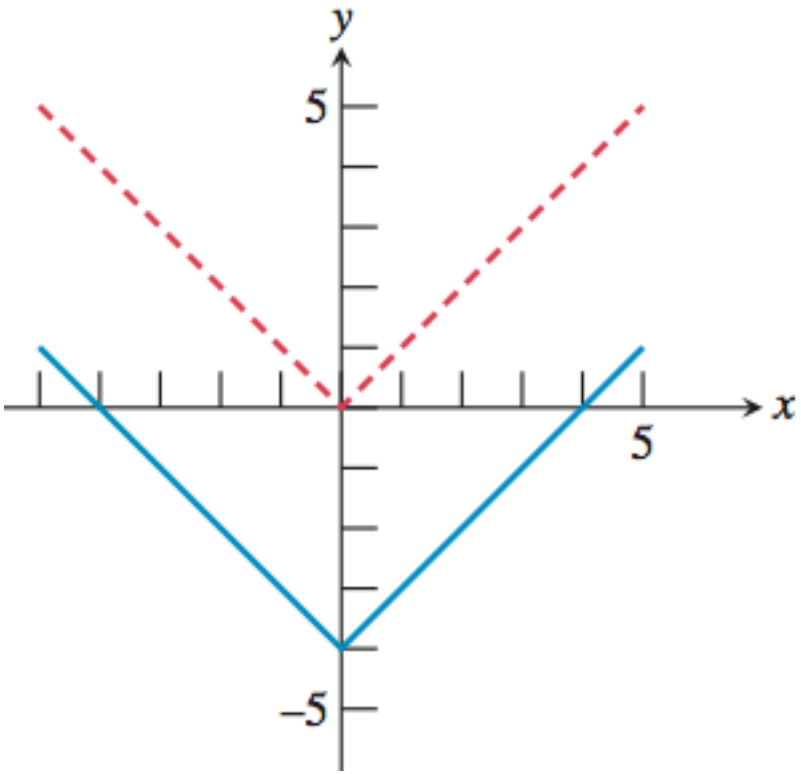
Example **Vertical Translations**

Describe how the graph of $f(x) = |x|$ can be transformed to the graph of $y = |x| - 4$.

Solution

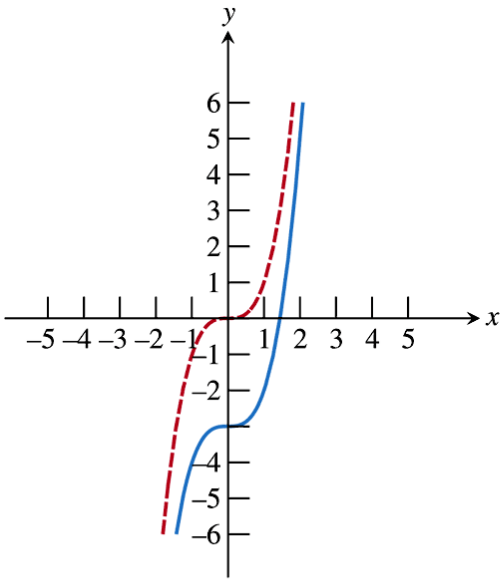
Describe how the graph of $f(x) = |x|$ can be transformed to the graph of $y = |x| - 4$.

It is a translation down by 4 units.

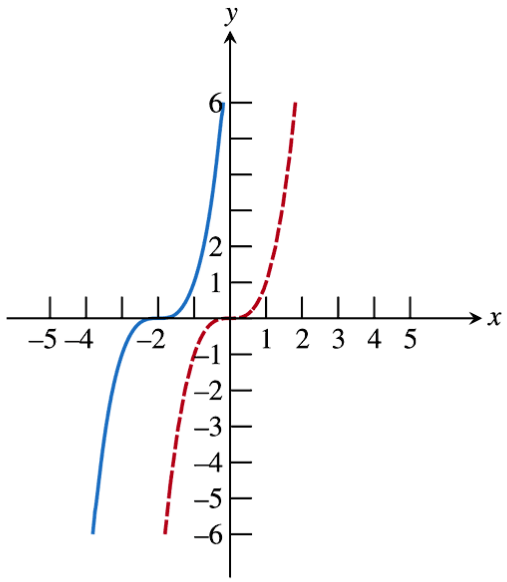


Example **Finding Equations for Translations**

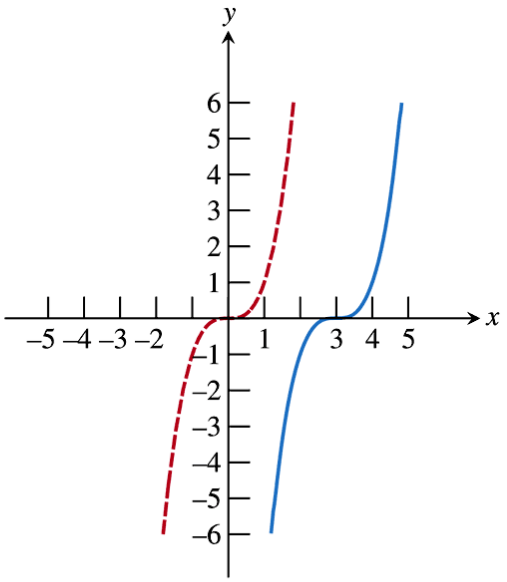
Each view shows the graph of $y_1 = x^3$ and a vertical or horizontal translation y_2. Write an equation for y_2.



(a)



(b)

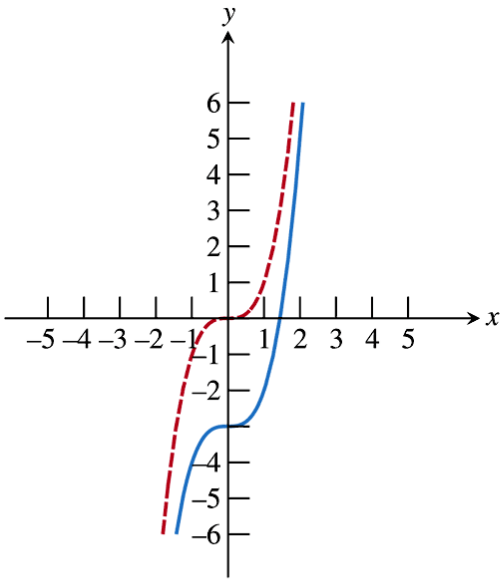


(c)

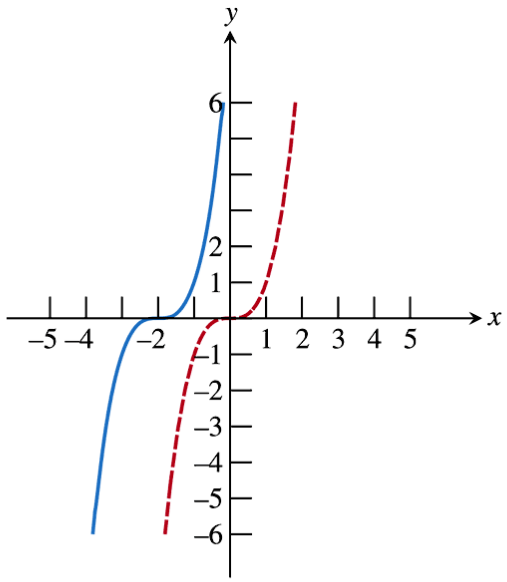
Solution

Each view shows the graph of $y_1 = x^3$ and a vertical or horizontal translation y_2. Write an equation for y_2.

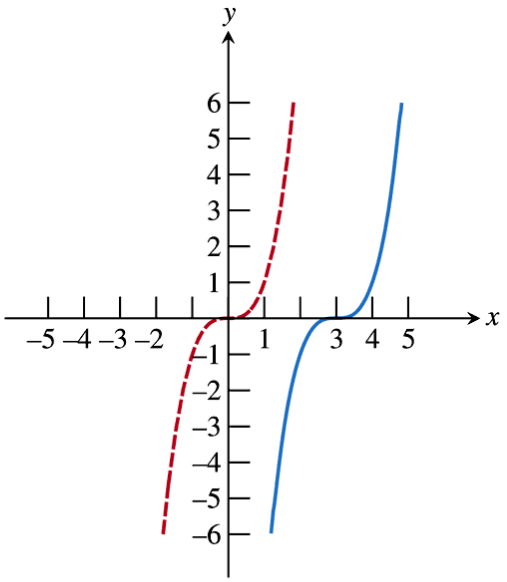
(a) $y_2 = x^3 - 3$ (b) $y_2 = (x+2)^3$ (c) $y_2 = (x-3)^3$



(a)



(b)



(c)

Reflections

The following transformations result in reflections of the graph of $y = f(x)$:

Across the x-axis

$$y = -f(x)$$

Across the y-axis

$$y = f(-x)$$

Graphing Absolute Value Compositions

Given the graph of $y = f(x)$,

the graph $y = |f(x)|$ can be obtained by reflecting the portion of the graph below the x-axis across the x-axis, leaving the portion above the x-axis unchanged;

the graph of $y = f(|x|)$ can be obtained by replacing the portion of the graph to the left of the y-axis by a reflection of the portion to the right of the y-axis across the y-axis, leaving the portion to the right of the y-axis unchanged. (The result will show even symmetry.)

Stretches and Shrinks

Let c be a positive real number. Then the following transformations result in stretches or shrinks of the graph of $y = f(x)$:

Horizontal Stretches or Shrinks

$$y = f\left(\frac{x}{c}\right) \qquad \begin{cases} \text{a stretch by a factor of } c & \text{if } c > 1 \\ \text{a shrink by a factor of } c & \text{if } c < 1 \end{cases}$$

Vertical Stretches or Shrinks

$$y = c \cdot f(x) \qquad \begin{cases} \text{a stretch by a factor of } c & \text{if } c > 1 \\ \text{a shrink by a factor of } c & \text{if } c < 1 \end{cases}$$

Example **Finding Equations for Stretches and Shrinks**

Let C_1 be the curve defined by $y_1 = x^3 + 3$. Find equations for the following non-rigid transformations of C_1 :

(a) C_2 is a vertical stretch of C_1 by a factor of 4.

(b) C_3 is a horizontal shrink of C_1 by a factor of 1/3.

Solution

Let C_1 be the curve defined by $y_1 = x^3 + 3$. Find equations for the following non-rigid transformations of C_1 :

(a) C_2 is a vertical stretch of C_1 by a factor of 4.

(b) C_3 is a horizontal shrink of C_1 by a factor of 1/3.

$$\begin{aligned}(a)\ y_2 &= 4 \cdot f(x) \\ &= 4(x^3 + 3) \\ &= 4x^3 + 12\end{aligned}$$

$$\begin{aligned}(b)\ y_3 &= f\left(\frac{x}{1/3}\right) \\ &= f(3x) \\ &= (3x)^3 + 3 \\ &= 27x^3 + 3\end{aligned}$$

Example **Combining Transformations in Order**

The graph of $y = x^2$ undergoes the following transformations, in order.

Find the equation of the graph that results.

- a horizontal shift 5 units to the left
- a vertical stretch by a factor of 3
- a vertical translation 4 units up

Solution

The graph of $y = x^2$ undergoes the following transformations, in order.

Find the equation of the graph that results.

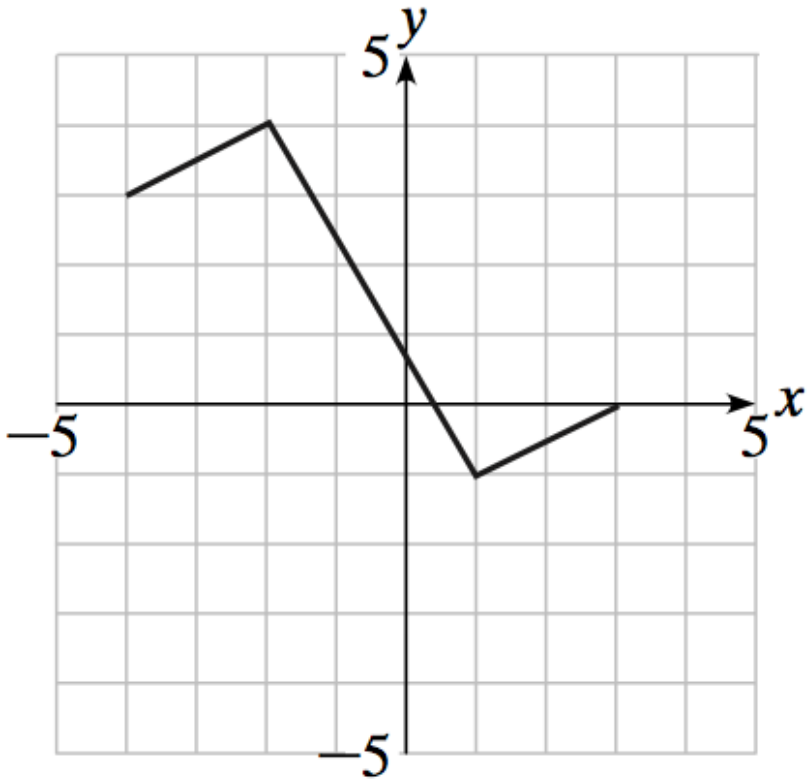
- a horizontal shift 5 units to the left
- a vertical stretch by a factor of 3
- a vertical translation 4 units up

$$x^2 \Rightarrow (x+5)^2 \Rightarrow 3(x+5)^2 \Rightarrow 3(x+5)^2 + 4$$

Expanding the final expression: $y = 3x^2 + 30x + 79$

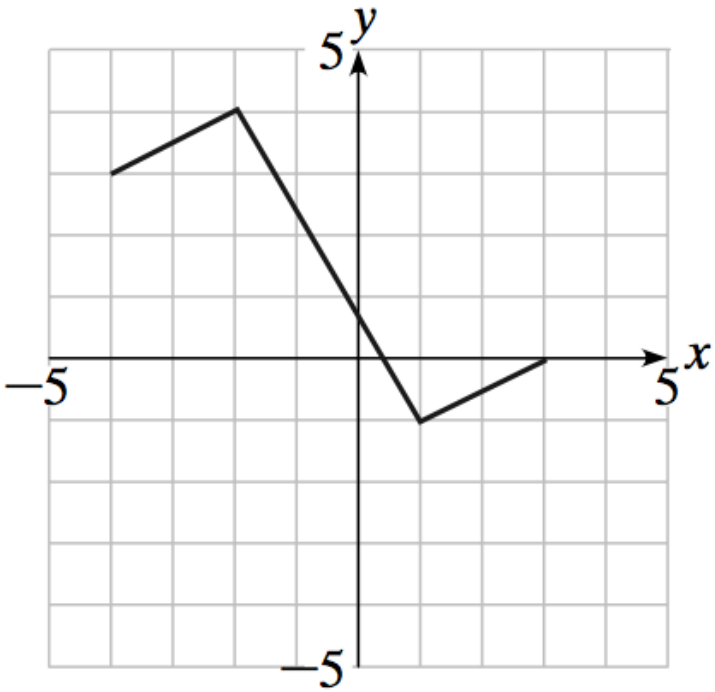
Example **Combining Transformations in Order**

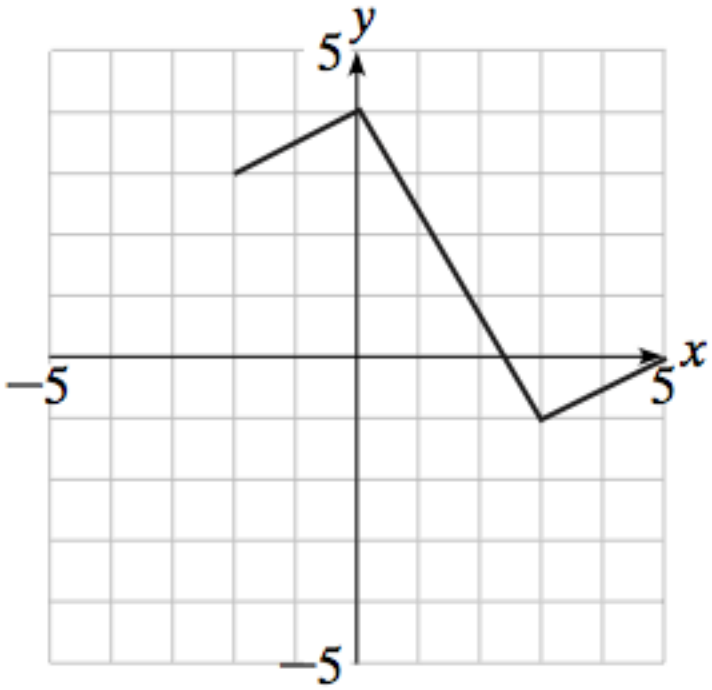
Describe how to transform the graph of $y = f(x)$ shown to the graph of $y = -f(x-2)+4$.



Solution

Describe how to transform the graph of $y = f(x)$ shown to the graph of $y = -f(x-2)+4$.

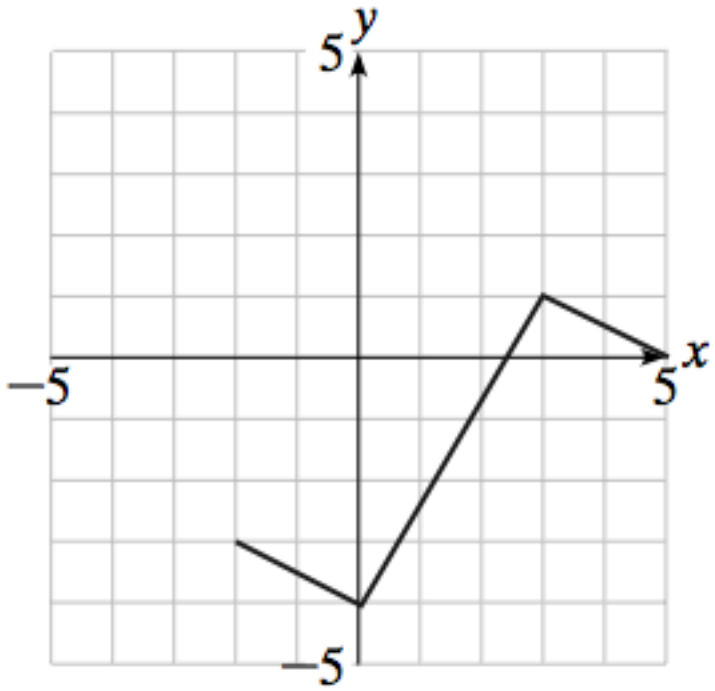




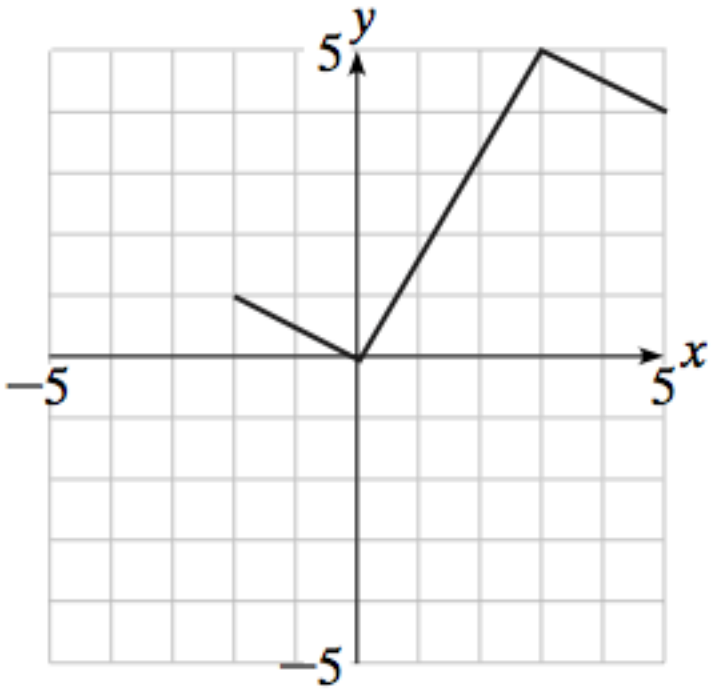
a. a horizontal shift right 2 units to get $y = f(x-2)$

Solution (continued)

Describe how to transform the graph of $y = f(x)$ shown to the graph of $y = -f(x-2)+4$.



b. a reflection over the x-axis to get $y = -f(x-2)$



c. a vertical shift up 4 units to get $y = -f(x-2)+4$

Quick Review

Write the expression as a binomial squared.

1. $x^2 + 4x + 4$

2. $x^2 - 2x + 1$

3. $4x^2 + 36x + 81$

Perform the indicated operations and simplify.

4. $(x-1)^2 + (x-1) + 2$

5. $(x-1)^3 + (x-1) + 2$

Quick Review Solutions

Write the expression as a binomial squared.

1. $x^2 + 4x + 4$ $(x+2)^2$

2. $x^2 - 2x + 1$ $(x-1)^2$

3. $4x^2 + 36x + 81$ $(2x+9)^2$

Perform the indicated operations and simplify.

4. $(x-1)^2 + (x-1) + 2$ $x^2 - x + 2$

5. $(x-1)^3 + (x-1) + 2$ $x^3 - 3x^2 + 4x$