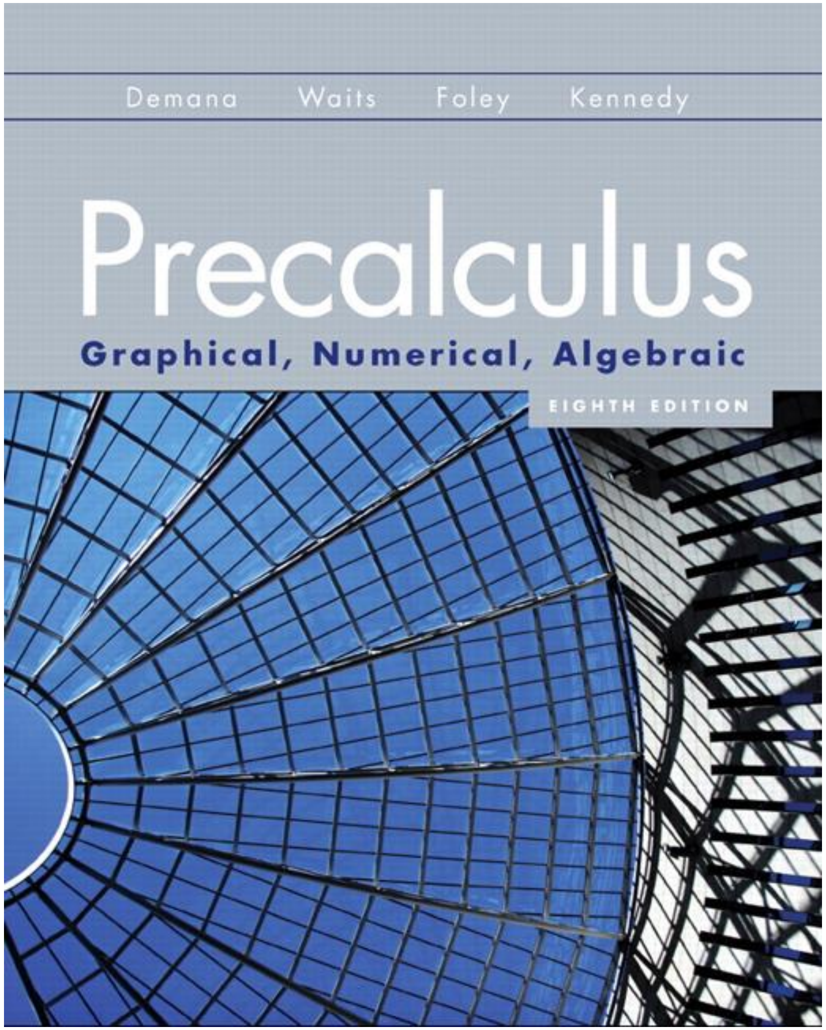
1.5

Parametric Relations and Inverses



What you'll learn about

- Relations Parametrically
- Inverse Relations and Inverse Functions

… and why

Some functions and graphs can best be defined parametrically, while some others can be best understood as inverses of functions we already know.

Relations Defined Parametrically

Another natural way to define functions or, more generally, relations, is to define *both* elements of the ordered pair (x, y) in terms of another variable t, called a **parameter**.

Example **Defining a Function Parametrically**

Consider the set of all ordered pairs (x, y) defined by the equations

$$x = t - 1$$

$$y = t^2 + 2$$

Find the points determined by $t = -3, -2, -1, 0, 1, 2, 3.$

Solution

Consider all ordered pairs (x, y) defined by

$x = t - 1$ and $y = t^2 + 2$

Find the points determined by $t = -3, -2, -1, 0, 1, 2, 3.$

t	$x = t - 1$	$y = t^2 + 2$	(x, y)
–3	–4	11	(–4, 11)
–2	–3	6	(–3, 6)
–1	–2	3	(–2, 3)
0	–1	2	(–1, 2)
1	0	3	(0, 3)
2	1	6	(1, 6)
3	2	11	(2, 11)

Example **Defining a Function Parametrically**

Consider the set of all ordered pairs (x, y) defined by the equations

$$x = t - 1$$

$$y = t^2 + 2$$

Find an algebraic relationship between x and y. Is y a function of x?

Solution

$x = t - 1$ and $y = t^2 + 2$

Find an algebraic relationship between x and y. Is y a function of x?

$y = t^2 + 2$

$y = (x+1)^2 + 2$ Solve for t in terms of x

$y = x^2 + 2x + 3$ Expand and simplify

Yes, y is a function of x.

Inverse Relation

The ordered pair (a,b) is in a relation if and only if the pair (b,a) is in the **inverse relation**.

Horizontal Line Test

The inverse of a relation is a function if and only if each horizontal line intersects the graph of the original relation in at most one point.

Inverse Function

If f is a one-to-one function with domain D and range R, then the **inverse function of** $\boldsymbol{f}$, denoted f^{-1}, is the function with domain R and range D defined by

$$f^{-1}(b) = a \text{ if and only if } f(a) = b.$$

Example **Finding an Inverse Function Algebraically**

Find an equation for $f^{-1}(x)$ if $f(x)=\dfrac{2x}{x-1}$.

Solution

Find an equation for $f^{-1}(x)$ if $f(x) = \dfrac{2x}{x-1}$.

$x = \dfrac{2y}{y-1}$ Switch the x and y

Solve for y:

$x(y-1) = 2y$ Multiply by $y-1$

$xy - x = 2y$ Distribute x

$xy - 2y = x$ Isolate the y terms

$y(x-2) = x$ Factor out y

$y = \dfrac{x}{x-2}$ Divide by $x-2$

Therefore

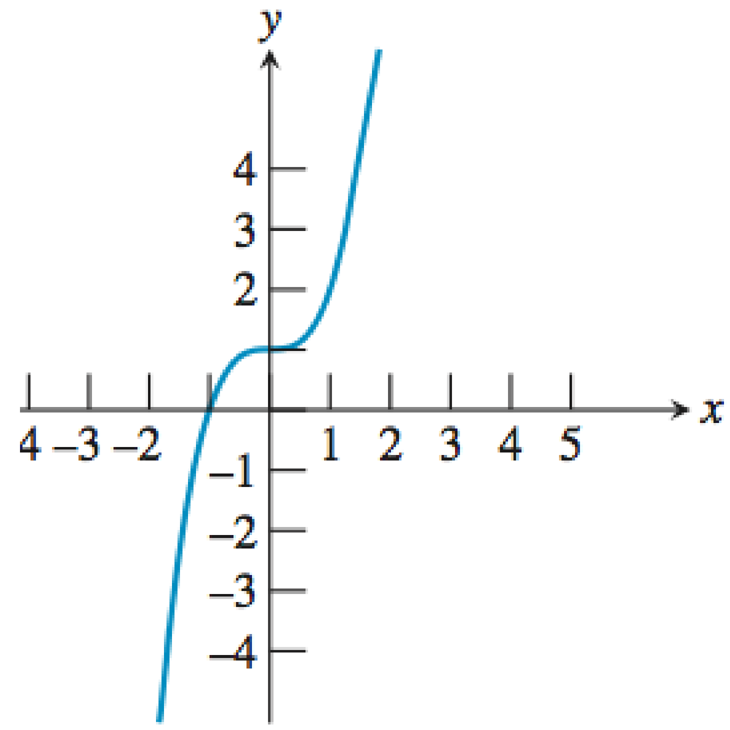
$$f^{-1}(x) = \frac{x}{x-2}.$$

The Inverse Reflection Principle

The points (a, b) and (b, a) in the coordinate plane are symmetric with respect to the line $y = x$. The points (a, b) and (b, a) are **reflections** of each other across the line $y = x$.

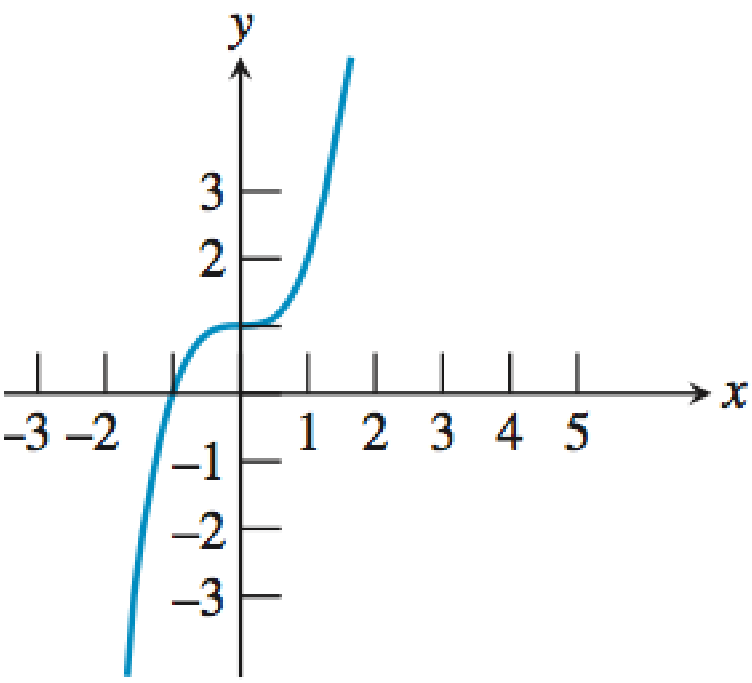
Example Finding an Inverse Function Graphically

The graph of a function $y = f(x)$ is shown. Sketch a graph of the function $y = f^{-1}(x)$. Is f a one-to-one function?

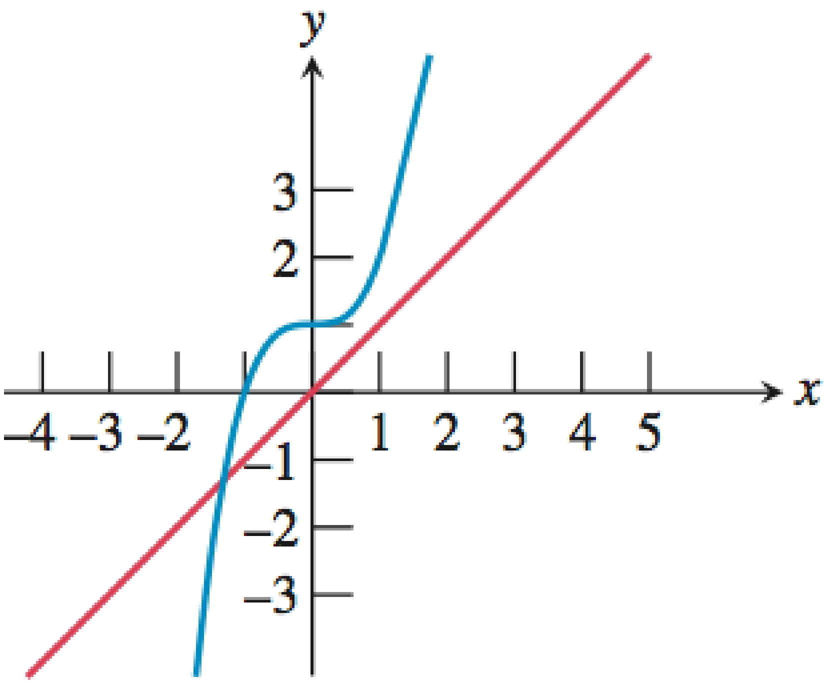


Solution

All we need to do is to find the reflection of the given graph across the line $y = x$.



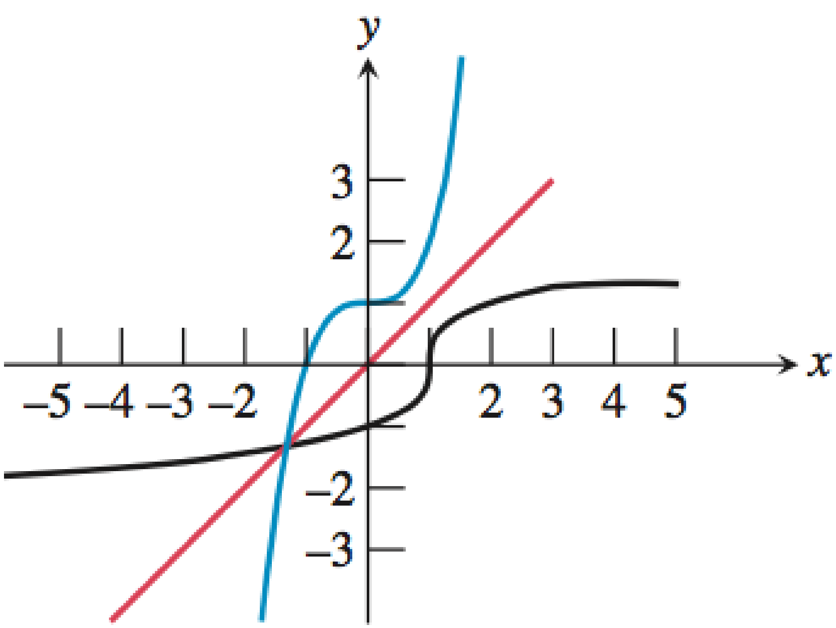
The graph of f.



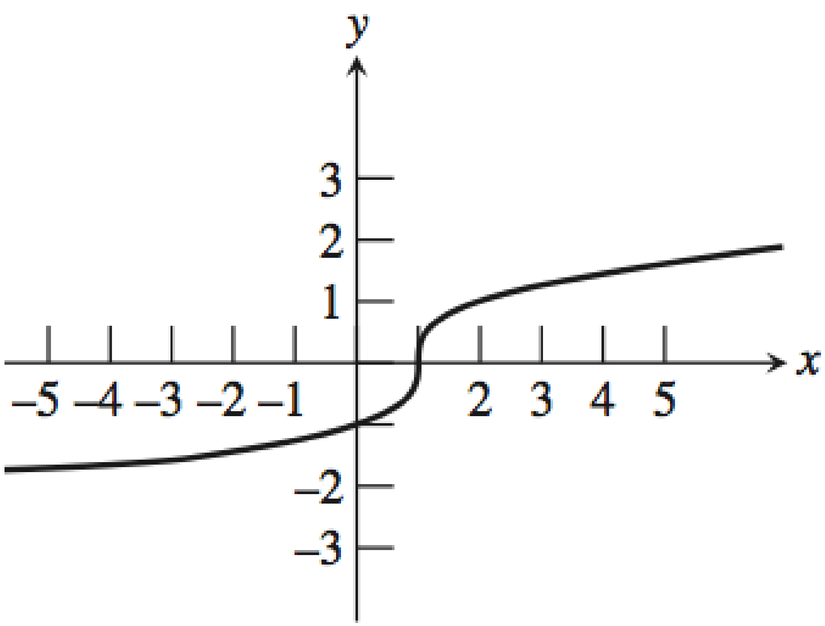
The mirror $y = x$.

Solution

All we need to do is to find the reflection of the given graph across the line $y = x$.



The reflection.



The graph of f^{-1}.

The Inverse Composition Rule

A function f is one-to-one with inverse function g if and only if $f(g(x)) = x$ for every x in the domain of g, and $g(f(x)) = x$ for every x in the domain of f.

Example **Verifying Inverse Functions**

Show algebraically the $f(x) = x^3 + 2$

and $g(x) = \sqrt[3]{x-2}$ are inverse functions.

Solution

Show algebraically the $f(x) = x^3 + 2$

and $g(x) = \sqrt[3]{x-2}$ are inverse functions.

Use the Inverse Composition Rule:

$$f(g(x)) = f(\sqrt[3]{x-2}) = \left(\sqrt[3]{x-2}\right)^3 + 2 = x - 2 + 2 = x$$

$$g(f(x)) = g(x^3 + 2) = \sqrt[3]{\left(x^3 + 2\right) - 2} = \sqrt[3]{x^3} = x$$

Since these equations are true for all x, f and g are inverses.

How to Find an Inverse Function Algebraically

Given a formula for a function f, proceed as follows to find a formula for f^{-1}.

1. Determine that there is a function f^{-1} by checking that f is one-to-one.
   State any restrictions on the domain of f.
2. Switch x and y in the formula $y = f(x)$.
3. Solve for y to get the formula for $y = f^{-1}(x)$.
   State any restrictions on domain of f^{-1}.

Quick Review

Solve the equation for y.

1. $x = 0.1y + 10$
2. $x = y^2 - 1$
3. $x = \dfrac{3}{y+2}$
4. $x = \dfrac{y+1}{y+2}$
5. $x = \sqrt{y+2},\ y \geq -2$

Quick Review Solutions

Solve the equation for *y*.

1. $x = 0.1y + 10$ $\quad y = 10x - 100$

2. $x = y^2 - 1$ $\quad y = \pm\sqrt{x+1}$

3. $x = \dfrac{3}{y+2}$ $\quad y = \dfrac{3}{x} - 2$

4. $x = \dfrac{y+1}{y+2}$ $\quad y = \dfrac{1-2x}{x-1}$

5. $x = \sqrt{y+2},\ y \geq -2$ $\quad y = x^2 - 2,\ y \geq -2$ and $x \geq 0$