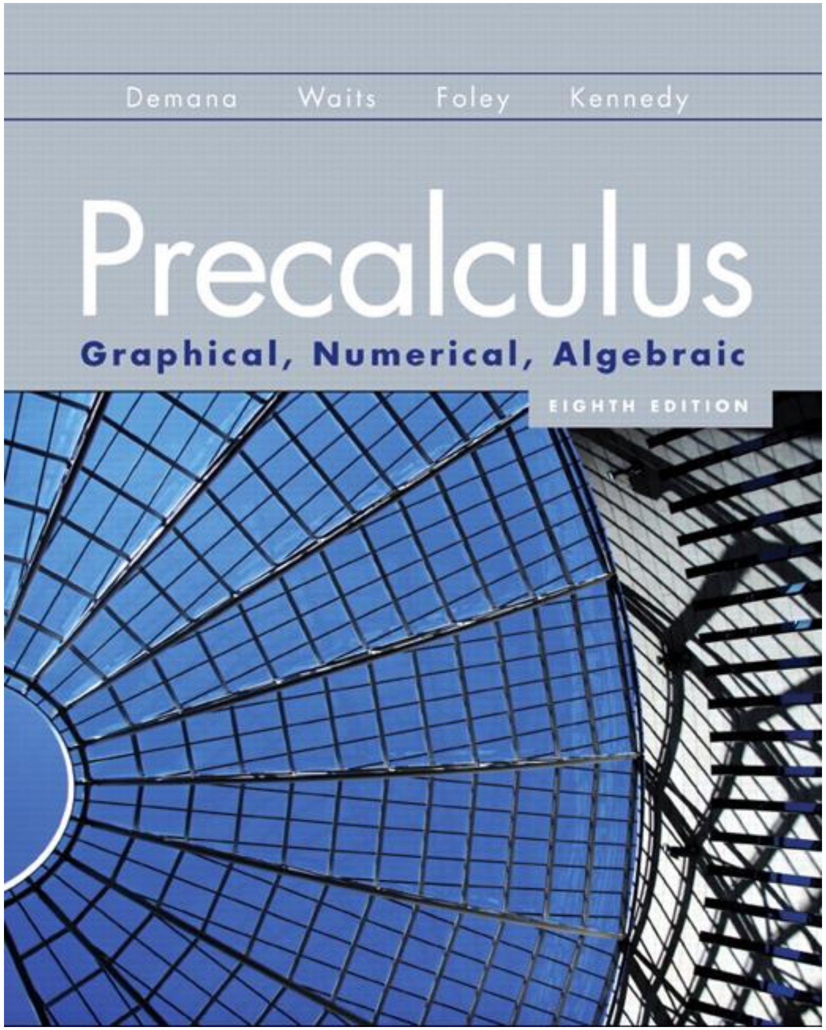
1.4

Building Functions from Functions



What you'll learn about

- Combining Functions Algebraically
- Composition of Functions
- Relations and Implicitly Defined Functions

… and why

Most of the functions that you will encounter in calculus and in real life can be created by combining or modifying other functions.

Sum, Difference, Product, and Quotient

Let f and g be two functions with intersecting domains. Then for all values of x in the intersection, the algebraic combinations of f and g are defined by the following rules:

Sum: $(f+g)(x) = f(x) + g(x)$

Difference: $(f-g)(x) = f(x) - g(x)$

Product: $(fg)(x) = f(x)g(x)$

Quotient: $\left(\dfrac{f}{g}\right)(x) = \dfrac{f(x)}{g(x)}$, provided $g(x) \neq 0$

In each case, the domain of the new function consists of all numbers that belong to both the domain of f and the domain of g.

Example **Defining New Functions Algebraically**

Let $f(x) = x^3$ and $g(x) = \sqrt{x+1}$.

Find formulas of the functions

(a) $f + g$

(b) $f - g$

(c) fg

(d) f / g

Solution

Let $f(x) = x^3$ and $g(x) = \sqrt{x+1}$.

Find formulas of the functions

(a) $f(x) + g(x) = x^3 + \sqrt{x+1}$ with domain $[-1, \infty)$

(b) $f(x) - g(x) = x^3 - \sqrt{x+1}$ with domain $[-1, \infty)$

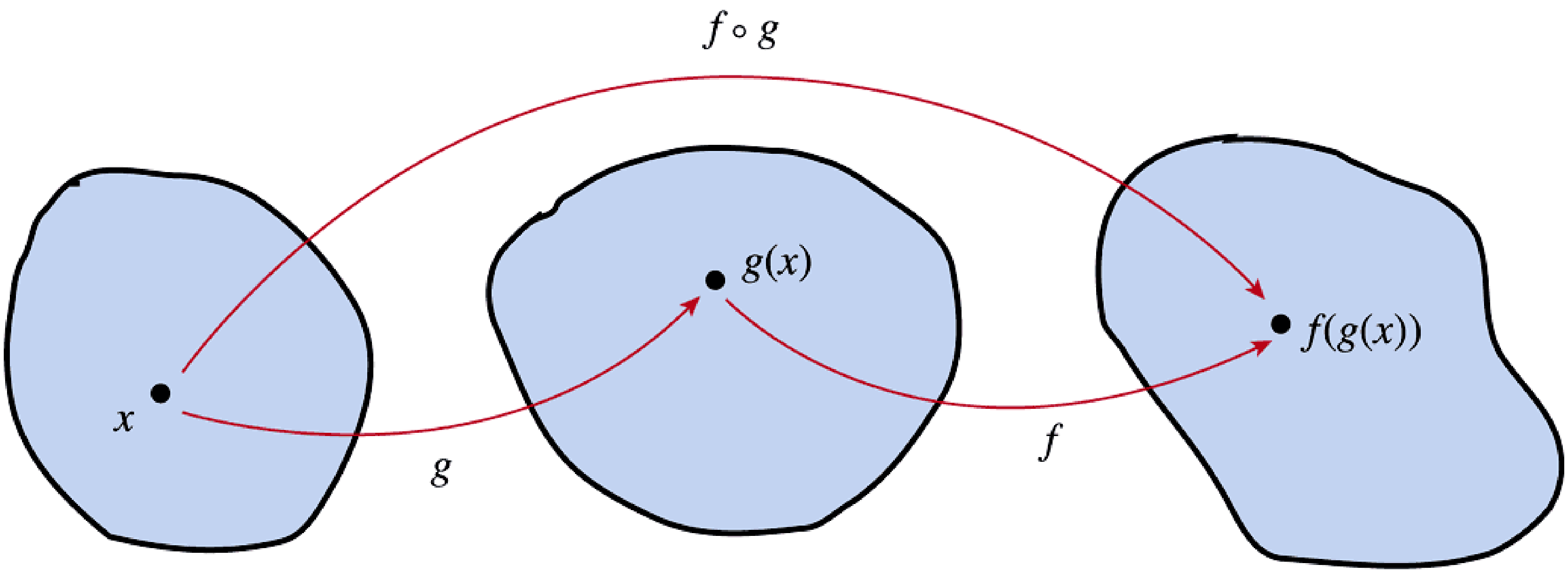
(c) $f(x)g(x) = x^3\sqrt{x+1}$ with domain $[-1, \infty)$

(d) $\dfrac{f(x)}{g(x)} = \dfrac{x^3}{\sqrt{x+1}}$ with domain $(-1, \infty)$

Composition of Functions

Let f and g be two functions such that the domain of f intersects the range of g. The composition f of g, denoted $f \circ g$, is defined by the rule $(f \circ g)(x) = f(g(x))$. The domain of $f \circ g$ consists of all x-values in the domain of g that map to $g(x)$-values in the domain of f.

Composition of Functions



x must be in the domain of *g* and *g*(*x*) must be in the domain of *f*

Example **Composing Functions**

Let $f(x)=2^{x}$ and $g(x)=\sqrt{x+1}$. Find

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

Solution

Let $f(x) = 2^x$ and $g(x) = \sqrt{x+1}$. Find

(a) $(f \circ g)(x)$

(b) $(g \circ f)(x)$

(a) $(f \circ g)(x) = f(g(x)) = 2^{\sqrt{x+1}}$

(b) $(g \circ f)(x) = g(f(x)) = \sqrt{2^x + 1}$

Example **Decomposing Functions**

Find f and g such that $h(x) = f(g(x))$.

$$h(x) = \sqrt{x^2 + 5}$$

Solution

Find f and g such that $h(x) = f(g(x))$.

$$h(x) = \sqrt{x^2 + 5}$$

One possible decomposition:

$$f(x) = \sqrt{x} \text{ and } g(x) = x^2 + 5$$

Another possibility:

$$f(x) = \sqrt{x+5} \text{ and } g(x) = x^2$$

Implicitly Defined Functions

The general term for a set of ordered pairs (x, y) is a **relation**. If the relation happens to relate a *single* value of y to each value of x, then the relation is also a function. In the case of $x^2 + y^2 = 4$, it is not a function itself, but we can split it into two equations that do define functions:

$$y = +\sqrt{4 - x^2} \quad \text{and} \quad y = -\sqrt{4 - x^2}$$

we say that the relation given by the equation defines the two functions.

Example Using Implicitly Defined Functions

Describe the graph of the relation $9x^2 + 4y^2 - 24y = 0$.

Solution

Describe the graph of the relation $9x^2 + 4y^2 - 24y = 0.$

$$9x^2 + 4y^2 - 24y = 0$$

$$9x^2 + 4\left(y^2 - 6y\right) = 0$$ Factor out 4

$$9x^2 + 4\left(y^2 - 6y + 9\right) = 0 + 4(9)$$ Complete the square

$$9x^2 + 4(y-3)^2 = 36$$

$$4(y-3)^2 = 36 - 9x^2$$ Solve for y

$$(y-3)^2 = \frac{36 - 9x^2}{4}$$

$$y - 3 = \frac{\pm\sqrt{36 - 9x^2}}{2}$$

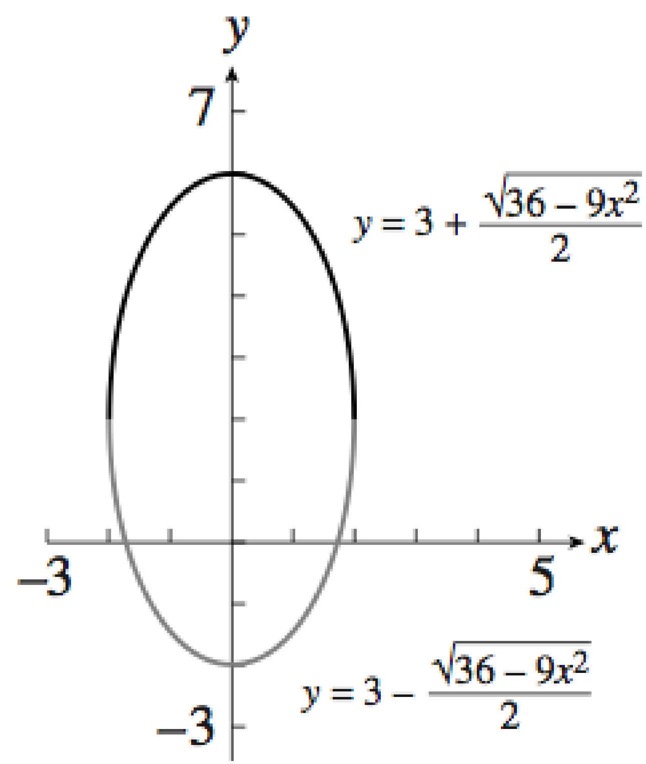
Solution (continued)

Describe the graph of the relation $9x^2 + 4y^2 - 24y = 0$.

$$y - 3 = \frac{\pm\sqrt{36 - 9x^2}}{2}$$

$$y = 3 + \frac{\sqrt{36 - 9x^2}}{2} \quad \text{and}$$

$$y = 3 - \frac{\sqrt{36 - 9x^2}}{2}$$



Quick Review

Find the domain of the function and express it in interval notation.

1. $f(x)=\dfrac{x+1}{x+4}$
2. $f(x)=\sqrt{x+1}$
3. $f(x)=\dfrac{1}{\sqrt{x+1}}$
4. $f(x)=\sqrt{\log x}$
5. $f(x)=4$

Quick Review Solutions

Find the domain of the function and express it in interval notation.

1. $f(x) = \dfrac{x+1}{x+4}$ $\quad (-\infty, -4) \cup (-4, \infty)$

2. $f(x) = \sqrt{x+1}$ $\quad [-1, \infty)$

3. $f(x) = \dfrac{1}{\sqrt{x+1}}$ $\quad (-1, \infty)$

4. $f(x) = \sqrt{\log x}$ $\quad (0, \infty)$

5. $f(x) = 4$ $\quad (-\infty, \infty)$