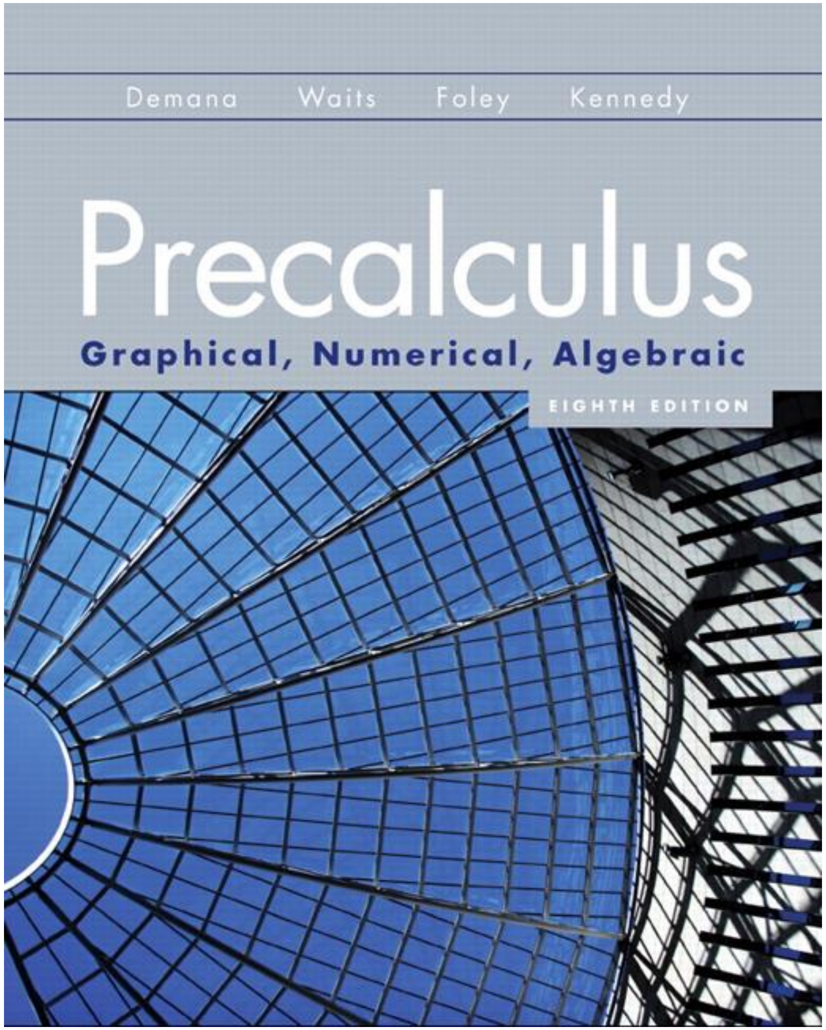
1.2

Functions and Their Properties



What you'll learn about

- Function Definition and Notation
- Domain and Range
- Continuity
- Increasing and Decreasing Functions
- Boundedness
- Local and Absolute Extrema
- Symmetry
- Asymptotes
- End Behavior

… and why

Functions and graphs form the basis for understanding the mathematics and applications you will see both in your work place and in coursework in college.

Function, Domain, and Range

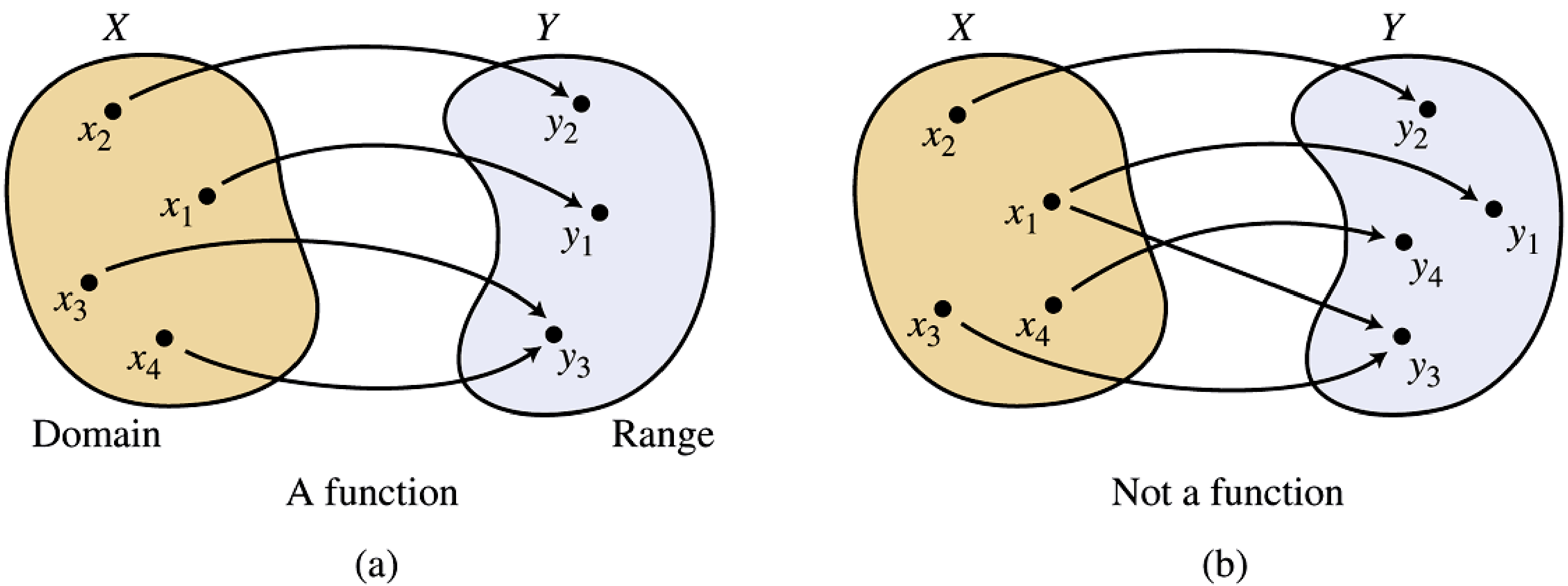
A function from a set D to a set R is a rule that assigns to every element in D a unique element in R. The set D of all input values is the **domain** of the function, and the set R of all output values is the **range** of the function.

Function Notation

To indicate that y comes from the function acting on x, we use Euler's elegant **function notation** $y = f(x)$ (which we read as "***y* equals *f* of *x***" or "**the value of *f* at *x***").

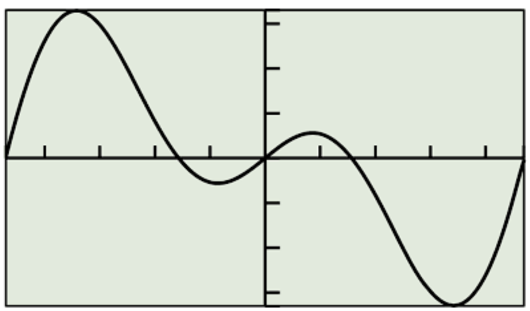
Here x is the **independent variable** and y is the **dependent variable**.

Mapping


X
Y
x_2
x_1
x_3
x_4
y_2
y_1
y_3
Domain
Range
A function
(a)
X
Y
x_2
x_1
x_3
x_4
y_2
y_1
y_4
y_3
Not a function
(b)

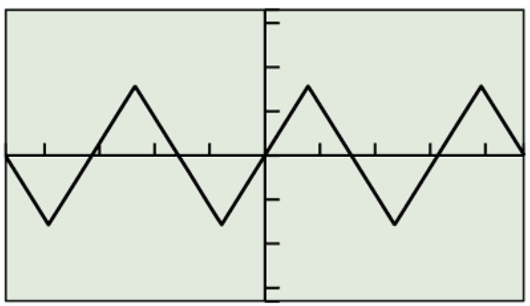
Example **Seeing a Function Graphically**

Of the three graphs shown below, which is not the graph of a function?



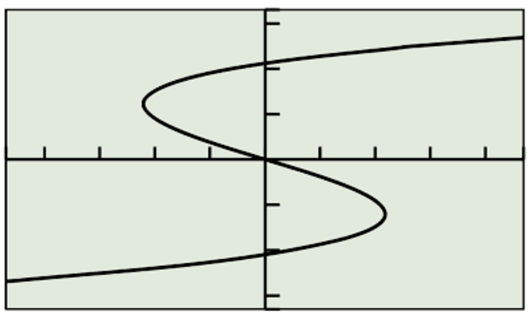
[–4.7, 4.7] by [–3.3, 3.3]

(a)



[–4.7, 4.7] by [–3.3, 3.3]

(b)

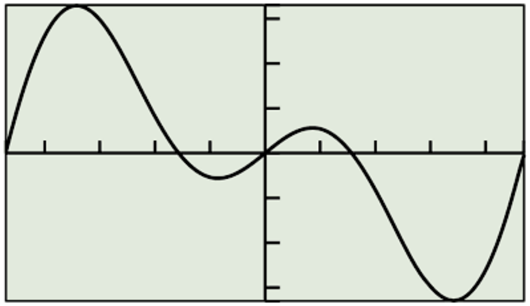


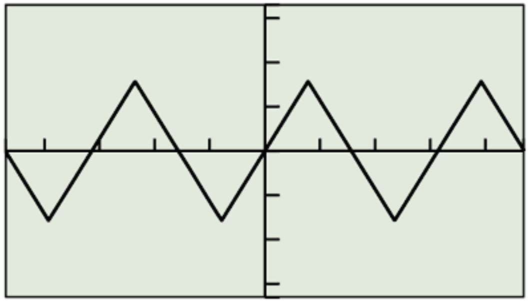
[–4.7, 4.7] by [–3.3, 3.3]

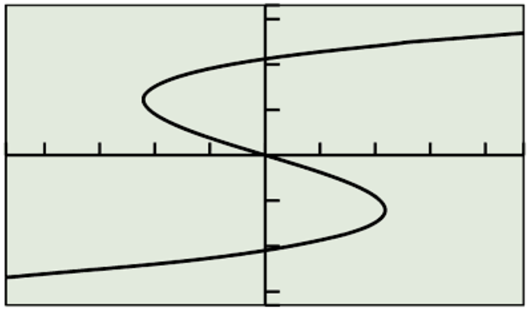
(c)

Solution

Of the three graphs shown below, which is not the graph of a function?


[–4.7, 4.7] by [–3.3, 3.3]
(a)


[–4.7, 4.7] by [–3.3, 3.3]
(b)


[–4.7, 4.7] by [–3.3, 3.3]
(c)

The graph in (c) is not the graph of a function. There are three points on the graph with *x*-coordinates 0.

Vertical Line Test

A graph (set of points (*x*,*y*)) in the *xy*-plane defines *y* as a function of *x* if and only if no vertical line intersects the graph in more than one point.

Agreement

Unless we are dealing with a model that necessitates a restricted domain, we will assume that the domain of a function defined by an algebraic expression is the same as the domain of the algebraic expression, the **implied domain**. For models, we will use a domain that fits the situation, the **relevant domain**.

Example **Finding the Domain of a Function**

Find the domain of the function.

$$f(x) = \sqrt{x+2}$$

Solution

Find the domain of the function.

$$f(x) = \sqrt{x+2}$$

Solve algebraically:

The expression under a radical may not be negative.

$$x + 2 \geq 0$$

$$x \geq -2$$

The domain of f is the interval $[-2, \infty)$.

Example **Finding the Range of a Function**

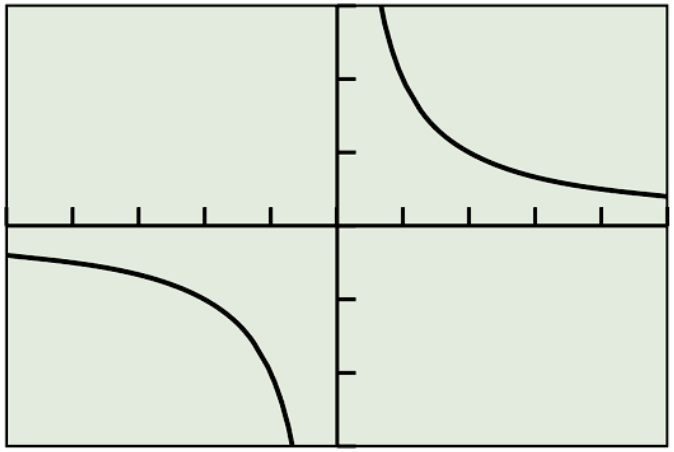
Find the range of the function $f(x) = \dfrac{2}{x}$.

Solution

Find the range of the function $f(x) = \dfrac{2}{x}$.

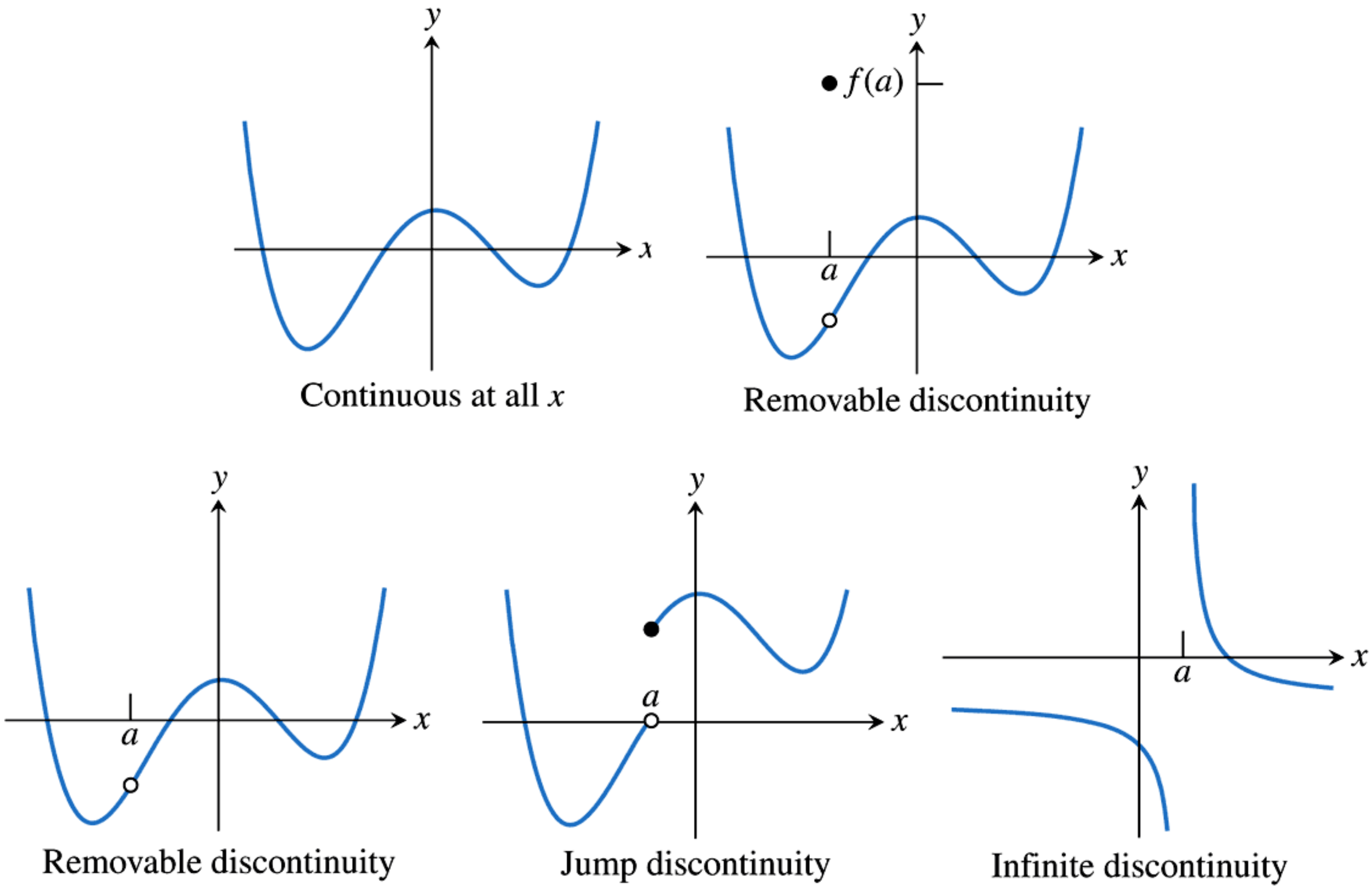
Solve Graphically:

The graph of $y = \dfrac{2}{x}$ shows that the range is all real numbers except 0. The range in interval notation is $(-\infty, 0) \cup (0, \infty)$.



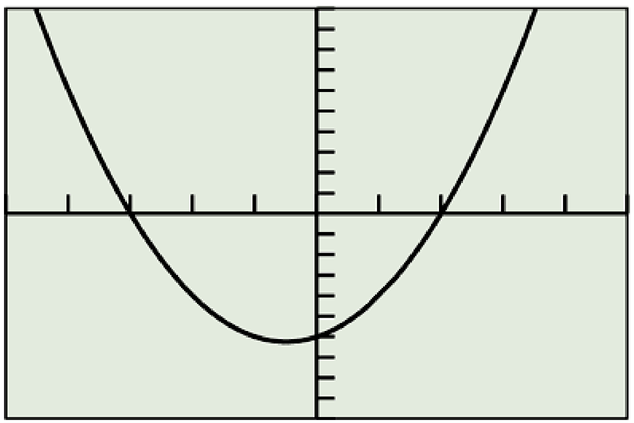
[–5, 5] by [–3, 3]

Continuity

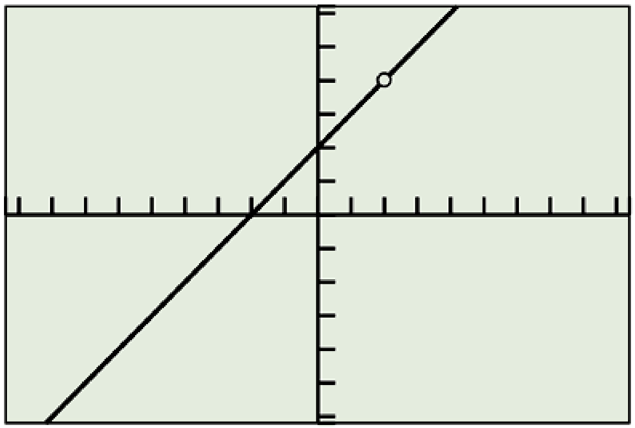

y
x
Continuous at all x
y
f(a)
a
x
Removable discontinuity
y
a
x
Removable discontinuity
y
a
x
Jump discontinuity
y
a
x
Infinite discontinuity

Example **Identifying Points of Discontinuity**

Which of the following figures shows functions that are discontinuous at $x = 2$?



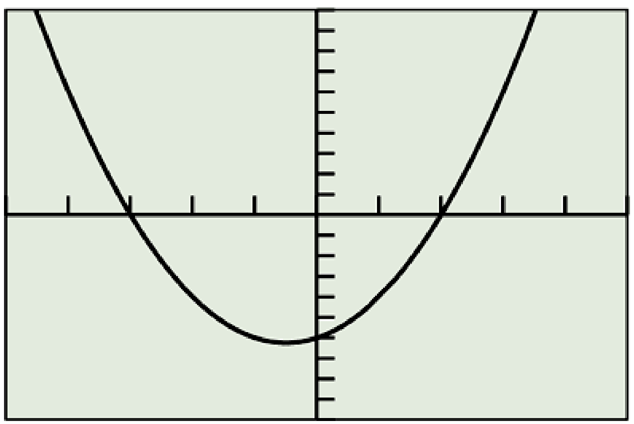
[–5, 5] by [–10, 10]



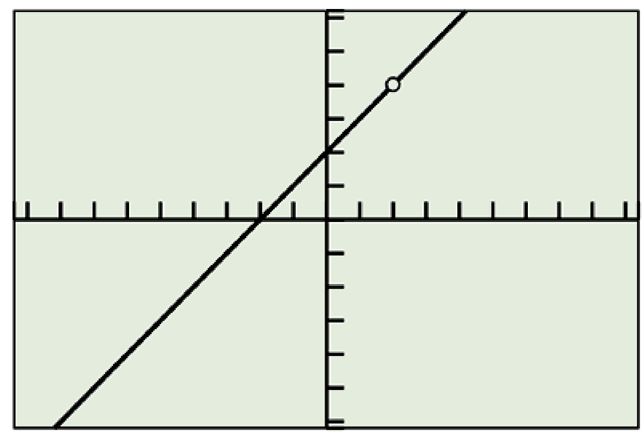
[–9.4, 9.4] by [–6.2, 6.2]

Solution

Which of the following figures shows functions that are discontinuous at $x = 2$?



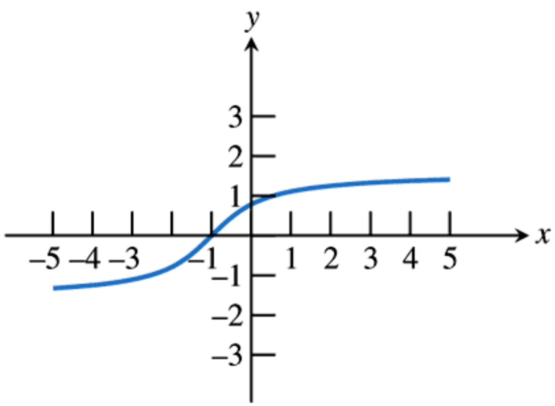
[–5, 5] by [–10, 10]



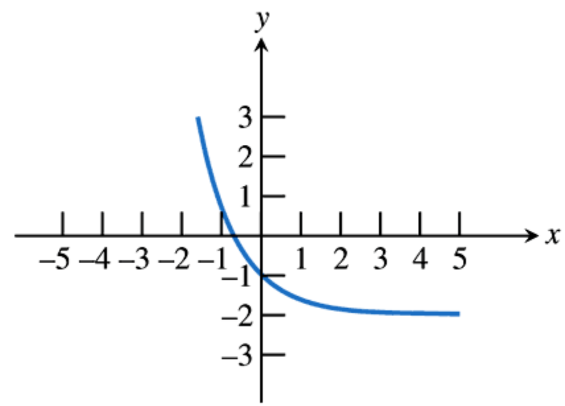
[–9.4, 9.4] by [–6.2, 6.2]

The function on the right is not defined at $x = 2$ and can not be continuous there. This is a removable discontinuity.

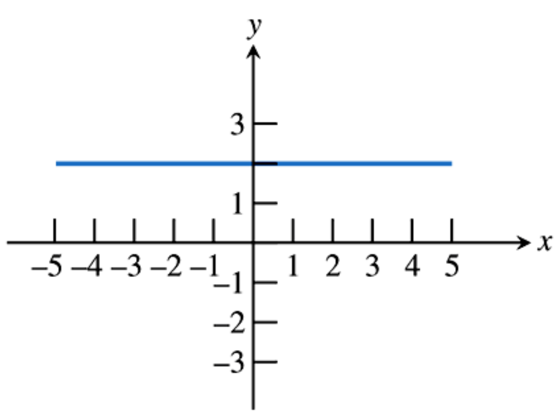
Increasing and Decreasing Functions



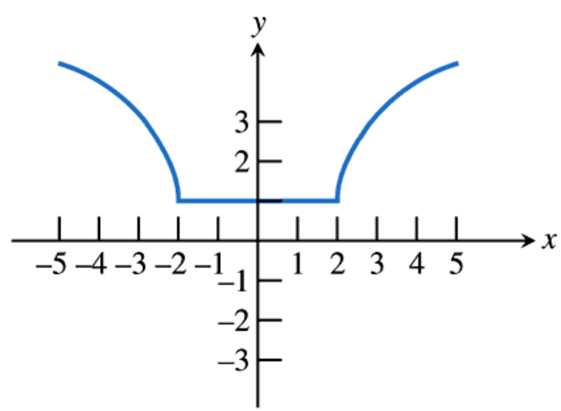
Increasing



Decreasing



Constant



Decreasing on (–∞, –2]
Constant on [–2, 2]
Increasing on [2, ∞)

Increasing, Decreasing, and Constant Function on an Interval

A function f is **increasing** on an interval if, for any two points in the interval, a positive change in x results in a positive change in $f(x)$.

A function f is **decreasing** on an interval if, for any two points in the interval, a positive change in x results in a negative change in $f(x)$.

A function f is **constant** on an interval if, for any two points in the interval, a positive change in x results in a zero change in $f(x)$.

Example **Analyzing a Function for Increasing-Decreasing Behavior**

Given $f(x) = |x+3| - |x-2| = \begin{cases} -5 & \text{if } x \le -3 \\ 2x+1 & \text{if } -3 < x < 2 \\ 5 & \text{if } x \ge 2 \end{cases}$

Identify the intervals on which $f(x)$ is increasing, decreasing and constant.

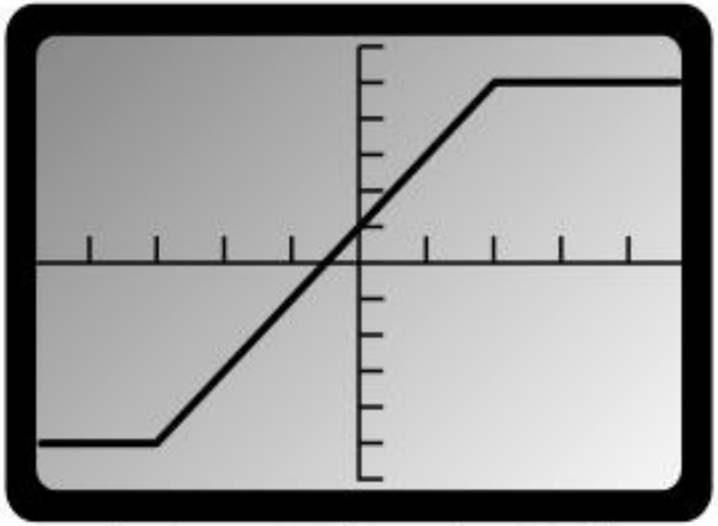
Solution

Given $f(x) = |x+3| - |x-2| = \begin{cases} -5 & \text{if } x \le -3 \\ 2x+1 & \text{if } -3 < x < 2 \\ 5 & \text{if } x \ge 2 \end{cases}$

Identify the intervals on which $f(x)$ is increasing, decreasing and constant.

The graph suggests, $f(x)$ is constant on $(-\infty, -3]$ and $(2, \infty)$.

On the interval $[-3, 2]$ $f(x)$ appears to be increasing.



[−4.7, 4.7] by [−6, 6]

Lower Bound, Upper Bound and Bounded

A function *f* is **bounded below** if there is some number *b* that is less than or equal to every number in the range of *f*. Any such number *b* is called a **lower bound** of *f*.

A function *f* is **bounded above** if there is some number *B* that is greater than or equal to every number in the range of *f*. Any such number *B* is called a **upper bound** of *f*.

A function *f* is **bounded** if it is bounded both above and below.

Local and Absolute Extrema

A **local maximum** of a function f is a value $f(c)$ that is greater than or equal to all range values of f on some open interval containing c. If $f(c)$ is greater than or equal to all range values of f, then $f(c)$ is the **maximum** (or **absolute maximum**) value of f.

A **local minimum** of a function f is a value $f(c)$ that is less than or equal to all range values of f on some open interval containing c. If $f(c)$ is less than or equal to all range values of f, then $f(c)$ is the **minimum** (or **absolute minimum**) value of f.

Local extrema are also called **relative extrema**.

Example **Identifying Local Extrema**

Decide whether $f(x) = x^4 - 7x^2 + 6x$ has any local maxima or minima. If so, find each local maximum value or minimum value and the value at which each occurs.

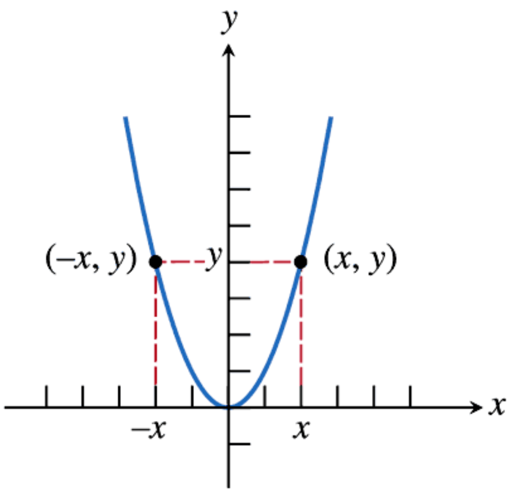
Solution

The graph of $f(x) = x^4 - 7x^2 + 6x$ suggests that there are two local maximum values and one local minimum value. We use the graphing calculator to approximate local minima as –24.06 (which occurs at $x < -2.06$) and –1.77 (which occurs at $x > 1.60$). Similarly, we identify the (approximate) local maximum as 1.32 (which occurs at $x > 0.46$).

Symmetry with respect to the *y*-axis

Example: $f(x) = x^2$

Graphically



Numerically

x	$f(x)$
-3	9
-2	4
-1	1
1	1
2	4
3	9

Algebraically

For all x in the domain of f,

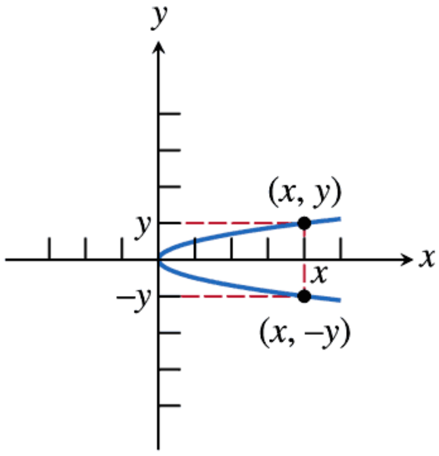
$$f(-x) = f(x)$$

Functions with this property (for example, x^n, n even) are **even** functions.

Symmetry with respect to the *x*-axis

Example: $x = y^2$

Graphically



Numerically

x	y
9	−3
4	−2
1	−1
1	1
4	2
9	3

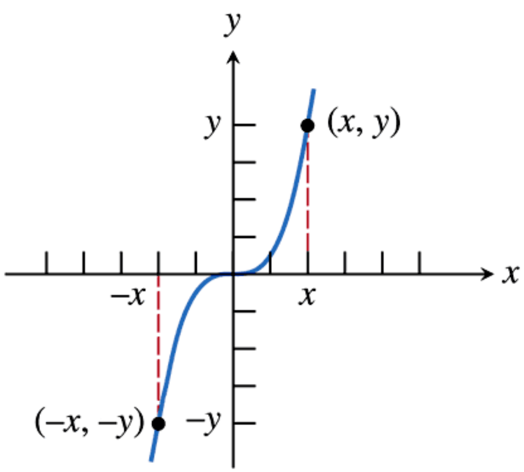
Algebraically

Graphs with this kind of symmetry are not functions (except the zero function), but we can say that $(x, -y)$ is on the graph whenever (x, y) is on the graph.

Symmetry with respect to the origin

Example: $f(x) = x^3$

Graphically



Numerically

x	y
-3	-27
-2	-8
-1	-1
1	1
2	8
3	27

Algebraically

For all x in the domain of f,

$$f(-x) = -f(x).$$

Functions with this property (for example, x^n, n odd) are **odd** functions.

Example **Checking Functions for Symmetry**

Tell whether the following function is odd, even, or neither.

$$f(x) = x^2 + 3$$

Solution

Tell whether the following function is odd, even, or neither.

$f(x) = x^2 + 3$

Solve Algebraically:

Find $f(-x)$.

$$f(-x) = (-x)^2 + 3$$
$$= x^2 + 3$$
$$= f(x)$$

The function is even.

Horizontal and Vertical Asymptotes

The line $y = b$ is a horizontal asymptote of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of b as x approaches +∞ or -∞.

In limit notation: $\lim_{x \to -\infty} f(x) = b$ or $\lim_{x \to \infty} f(x) = b$.

The line $x = a$ is a vertical asymptote of the graph of a function $y = f(x)$ if $f(x)$ approaches a limit of +∞ or -∞ as x approaches a from either direction.

In limit notation: $\lim_{x \to a^-} f(x) = \pm\infty$ or $\lim_{x \to a^+} f(x) = \pm\infty$.

Example **Identifying the Asymptotes of a Graph**

□□□□□□□□□□□□□□□□□□□□□□□□□□□ □□□□□□□□ asymptotes of the graph of

$$y = \frac{x}{x^2 - x - 2}.$$

Solution

□□□□□□□□□□□□□□□□□□□□□□□□□□□
□□□□□□□□ asymptotes of the graph of $y = \dfrac{x}{x^2 - x - 2}$.

$y = \dfrac{x}{x^2 - x - 2}$ is undefined at $x = -1$ and $x = 2$,

These are the vertical asymptotes.

$$\lim_{x \to \infty} \frac{x}{x^2 - x - 2} . = 0$$

So $y = 0$ is a horizontal asymptote.

$$\lim_{x \to -\infty} \frac{x}{x^2 - x - 2} . = 0$$

Again $y = 0$ is a horizontal asymptote.

Quick Review

Solve the equation or inequality.

1. $x^2 - 9 < 0$

2. $x^2 - 16 = 0$

Find all values of x algebraically for which the algebraic expression is not defined.

3. $\dfrac{1}{x-3}$

4. $\sqrt{x-3}$

5. $\dfrac{\sqrt{x+1}}{\sqrt{x-3}}$

Quick Review Solutions

Solve the equation or inequality.

1. $x^2 - 9 < 0$ $\quad -3 < x < 3$

2. $x^2 - 16 = 0$ $\quad x = \pm 4$

Find all values of x algebraically for which the algebraic expression is not defined.

3. $\dfrac{1}{x-3}$ $\quad x = 3$

4. $\sqrt{x-3}$ $\quad x < 3$

5. $\dfrac{\sqrt{x+1}}{\sqrt{x-3}}$ $\quad x < 3$